Cal Poly Department of Mathematics

Puzzle of the Week

April 7 - April 20, 2016

A point is chosen (uniformly) at random inside an equilateral triangle. What is the probability that it lies closer to the center of the triangle than it does to any edge?

Solutions should be submitted to Morgan Sherman:

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before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in the next email announcement. Anybody associated to Cal Poly is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution:

The probability that the point is closer to the center is $\frac{5}{27} \approx 18.5\%$.

This is one of those problems whose statement is simple and clean, and whose answer is simple and clean, but whose solution seems to require some messy calculations (there may be a slick solution, I just don't happen to know one).

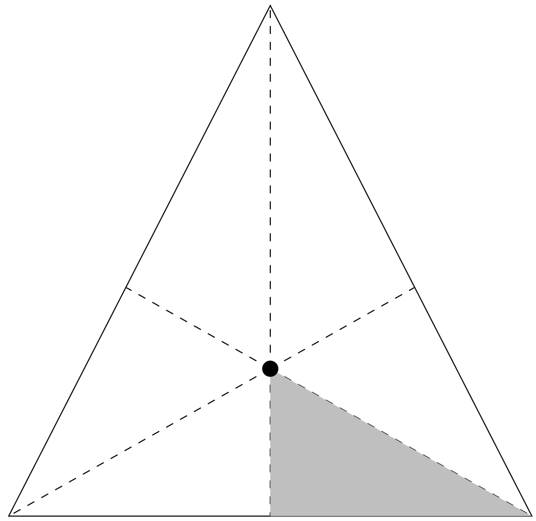
See the figure on the next page.

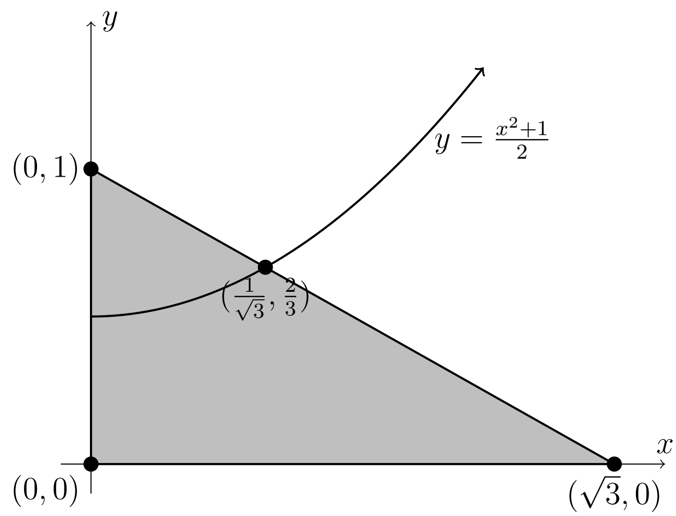
First of all the triangle is composed of six congruent smaller triangles, so by symmetry we may assume the chosen point lies in the one indicated in the figure. Now we need to calculate in this new triangle the ratio of area closer to the upper left vertex then the lower edge, to the area of the triangle itself.

Set up coordinates as indicated in the figure. The point (x, y) is equidistant from $(0, 1)$ as from the x-axis if

$$x^2 + (y-1)^2 = y, \quad \text{i.e. } y = \frac{x^2+1}{2}. \tag{1}$$

This parabola meets the left leg of the triangle at the point $(0, \frac{1}{2})$, while it meets the hypotenuse





at the point $(\frac{1}{\sqrt{3}}, \frac{2}{3})$. Hence the area we are interested in can be calculated as

$$A_C = \int_0^{\frac{1}{\sqrt{3}}} \left(\left(1 - \frac{x}{\sqrt{3}} \right) - \left(\frac{x^2+1}{2} \right) \right) dx = \frac{5}{18\sqrt{3}}. \tag{2}$$

Whereas the area of the triangle is $A_T = \frac{\sqrt{3}}{2}$. Hence the probability is

$$A_C/A_T = \left(\frac{5}{18\sqrt{3}} \right) / \left(\frac{\sqrt{3}}{2} \right) = \frac{5 \cdot 2}{18 \cdot 3} = \frac{5}{27}.$$