Cal Poly Department of Mathematics

Puzzle of the Week

Oct 2 - 8, 2014

A sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ satisfies

$$a_{n+3} = a_{n+2}(a_{n+1} + a_n),\ n = 1, 2, 3, \ldots$$

If $a_6 = 8820$, determine the possible values of $a_1, a_2, a_3, a_7,$ and a_8.

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: There are two possibilities: $(a_1, a_2, a_3) = (2, 2, 7)$ or $(29, 6, 1)$.

We first calculate that $8820 = 2^2 \cdot 3^2 \cdot 5 \cdot 7^2$ and, using the recurrence relation repeatedly, we get $a_6 = a_3^2(a_3 + a_2)(a_2 + a_1)(a_2 + a_1 + 1)$. We set $x = a_3, y = a_3 + a_2, z = a_2 + a_1$. Then we are looking for positive integer solutions to

$$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 = x^2 \cdot y \cdot z \cdot (z + 1), \qquad x < y < x + z. \tag{1}$$

We notice that one of z or $z + 1$ is even, so x cannot be, which leaves only four possibilities for $x^2:\ 1,\ 3^2,\ 7^2,\ 3^2 \cdot 7^2$.

If $x^2 = 1$: Then (1) implies that $8820 = y \cdot z \cdot (z + 1) < z(z + 1)^2$. It follows that $z > 20$. The only pair of divisors $z, z + 1$ to 8820 with $z > 20$ is $z = 5 \cdot 7 = 35$ and $z + 1 = 2^2 \cdot 3^2 = 36$. This leads to the solution $x = 1, y = 7, z = 35$, which gives $a_1 = 29, a_2 = 6, a_3 = 1$.

If $x^2 = 3^2$: Then (1) implies that $\frac{8820}{9} < z(z + 1)(z + 3)$. This implies $z > 8$. But there are no pairs of divisors $z, z + 1$ to $\frac{8820}{9} = 2^2 \cdot 5 \cdot 7^2$ with $z > 8$.

If $x^2 = 7^2$: Then (1) becomes $2^2 \cdot 3^2 \cdot 5 = yz(z + 1)$ and $7 < y < z + 7$. Also z and $z + 1$ must both divide $2^2 \cdot 3^2 \cdot 5$. A quick check and we find the only possibility is $(x, y, z) = (7, 9, 4)$, which leads to $a_1 = 2,\ a_2 = 2,\ a_3 = 7$.

If $x^2 = 3^2 \cdot 7^2$: Then (1) becomes $2^2 \cdot 5 = yz(z + 1), 21 < y < z + 21$. So $20 > y(y - 21)(y - 20)$ which gives $y < 1$, a contradiction.