Cal Poly Department of Mathematics

Puzzle of the Week

Feb 20-26, 2014

For every $r > 1$ calculate the value of

$$\sum_{n=0}^{\infty} \frac{2^n}{r^{2^n}+1} = \frac{1}{r+1} + \frac{2}{r^2+1} + \frac{4}{r^4+1} + \frac{8}{r^8+1} + \dots$$

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: The series sums to $1/(r-1)$.

The trick here is to rewrite each term as a geometric series, and then gather all terms with common powers of r:

$$\sum_{n\geq 0} \frac{2^n}{r^{2^n}+1} = \sum_{n\geq 0} \frac{2^n}{r^{2^n}} \frac{1}{1+r^{-2^n}} = \sum_{m,n\geq 0} \frac{2^n}{r^{2^n}} (-1)^m r^{-2^n m} = \sum_{m,n\geq 0} (-1)^m 2^n r^{-2^n(m+1)}$$

Now write $m+1 = 2^k(2\ell+1)$. Then the power of r^{-1} in the general term is $2^{k+n}(2\ell+1)$. So two such terms have a common power of r if and only if they share the same ℓ and the same value for $n+k$. Also note that $(-1)^m = -1 \iff k > 0$. Therefore we can rewrite the series:

$$\sum_{m,n\geq 0} (-1)^m 2^n r^{-2^n(m+1)} = \sum_{k',l\geq 0} \left[\sum_{n=0}^{k'} (-1)^m 2^n \right] r^{-2^{k'}(2\ell+1)} = \sum_{k,l\geq 0} \left[-\left(\sum_{n=0}^{k-1} 2^n \right) + 2^k \right] r^{-2^k(2\ell+1)}$$

$$= \sum_{k,l\geq 0} \left[-\frac{2^k-1}{2-1} + 2^k \right] r^{-2^k(2\ell+1)} = \sum_{k,l\geq 0} r^{-2^k(2\ell+1)} = \sum_{i=1}^{\infty} r^{-i} = \frac{r^{-1}}{1-r^{-1}}$$