Cal Poly Department of Mathematics

Puzzle of the Week

Jan 19-25, 2012

For a positive integer n let $B(n)$ denote the number of 1s in its binary representation. Find the value of

$$\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)}$$

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution:

The value of the sum is $\log 4$.

Let S denote the value of the sum. We note two key formulas: $B(2n) = B(n)$ and $B(2n+1) = 1 + B(2n) = 1 + B(n)$. Splitting the sum into odd and even terms we find:

$$\begin{aligned}
S &= \frac{B(1)}{1\cdot 2} + \sum_{n=1}^{\infty} \frac{B(2n)}{2n(2n+1)} + \sum_{n=1}^{\infty} \frac{B(2n+1)}{(2n+1)(2n+2)} \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{B(2n)}{2n} - \frac{B(2n)}{2n+1} + \frac{B(2n+1)}{2n+1} - \frac{B(2n+2)}{2n+1} \right) \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{B(n)}{2n} - \frac{B(n)}{2n+1} + \frac{1+B(n)}{2n+1} - \frac{1+B(n)}{2n+2} \right) \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+2} + \frac{1}{2}\frac{B(n)}{n(n+1)} \right) \\
&= \left((1 - \frac{1}{2}) + \frac{1}{3} - \frac{1}{4} + \dots \right) + \frac{1}{2}S
\end{aligned}$$

and the solution above follows.