Vector Fields are functions that map points in space to vectors. These types of functions occur frequently in electro-magnetic applications, forces moving an object, and many other fluid and gas flow problems. The integrals that result from these applications help the understanding and calculations of work, mass, flux, and other important elements in the behavior and design of Magnetic Resonant Imaging (MRI) devices, electronic chip making equipment, and many others.

This summary does not address the precise applicability of the Theorems presented. Consult an appropriate text to insure complete understanding of requirements. They typically include continuous functions with continuous partial derivatives, some over all of \( \mathbb{R}^n \). Additional restrictions may include piecewise smooth curves and open and/or union of simple regions.

**Definitions**

We use \( \vec{r} = (x, y, z) \), and \( \vec{p} = (x, y, z) \), for vectors/points in space with magnitude \(|\vec{r}| = \sqrt{x^2 + y^2 + z^2} \). We use parametric representations of a line/curve/path by \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \) and a surface by \( \vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \); thus \( f(\vec{r}) \) means the scalar function \( f \) evaluated at a point on the curve/surface. We define a vector field \( \vec{F} = \langle P, Q, R \rangle \) with \( P, Q, \) and \( R \) scalar functions of points in space.

Operators we will use are: 1) Del operator, a vector: \( \vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \), and Laplace operator \( \vec{\nabla} \cdot \vec{\nabla} = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \); 2) Gradient of \( f \): \( \vec{\nabla}f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \); 3) Curl of a vector \( \vec{F} = \langle P, Q, R \rangle \): \( \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \); 3) Divergence: \( \text{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \). Key identities: a) When a vector is ‘conservative,’ it is the gradient of a ‘potential’ function, and \( \text{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla} f = 0 \); b) Divergence of the curl of a vector is zero: \( \vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0 \).

**Line Integral of Scalar Functions** \( \int_C f(x, y, z) \, ds \)

\( \int_C f(\vec{r}) \, ds \) where \( C \) is a curve/line/path given by \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \ a \leq t \leq b \). When \( C \) is a straight line, a parametric representation is obtained using \( \vec{r}(t) = (1-t) \vec{r}_0 + t \vec{r}_1, \ 0 \leq t \leq 1 \), from two given points \( \vec{r}_0 \), and \( \vec{r}_1 \); when \( C \) is a curve like \( y = g(x), \ x_0 \leq x \leq x_1 \), then use \( \vec{r}(t) = \langle t, g(t), 0 \rangle, \ x_0 \leq t \leq x_1 \). Use polar-like equations to parameterize circular curves like \( \vec{r}(t) = \langle a \cos t, a \sin t, 0 \rangle, \ 0 \leq t \leq 2\pi \) for \( x^2 + y^2 = a^2 \). Then \( ds = | \vec{r}'(t) | \, dt \), with \( \vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \), and thus

\( \int_C f(\vec{r}) \, ds = \int_{x_0}^{x_1} \int_{\vec{r}(t)}^{\vec{r}(t)} f(\vec{r}(t)) | \vec{r}'(t) | \, dt \). Applications for these integrals for positive functions are the area of one side of a fence following curve \( C \) with height \( f \), or the mass of a wire with \( C \) shape and mass density \( f \).
**Line Integral of Vector Fields** $\int_{C} \vec{F} \cdot d\vec{r}$

Line integral of a Vector Field can be interpreted as the work done by the Vector Field Force moving an object along the path $C$. These integrals can be expressed by $\int_{C} \vec{F} \cdot d\vec{r} = \int_{C} (P \, dx + Q \, dy + R \, dz)$, and then using parameterized representation for path $C$ given by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$, the integral becomes

$$\int_{C} \vec{F} \cdot d\vec{r} = \int_{a}^{b} [P(\vec{r}(t)) \, x'(t) + Q(\vec{r}(t)) \, y'(t) + R(\vec{r}(t)) \, z'(t)] \, dt$$

**Line Integral of Gradient Vector Fields** $\int_{C} \nabla f \cdot d\vec{r}$

$$\int_{C} \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)),$$ where $a$ and $b$ define the end points of the curve/line/path $C$. This is the Fundamental Theorem of Calculus (FTC) for Line Integrals. If $C$ is a *closed* path $\int_{C} \nabla f \cdot d\vec{r} = 0$ since end points of path $C$ are the same.

**Potential Function $f$ given the Gradient Vector Field**

When a vector field is conservative, then it is equal to the gradient of some function; this function is called the ‘potential function’ for the vector field. Given this vector field we have $\vec{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$, and finding the potential function $f$ is extremely useful (as the above Line Integral of Gradient Vector Fields shows). We will use an example to show how to calculate $f$. Let $\vec{F} = \nabla f = \{4xe^z, \cos y, 2x^2e^z\}$. Begin with $f_x = 4xe^z \Rightarrow f = 2x^2e^z + g(y, z)$, but if we now take partial with respect to $y$ we have that $f_y = g_y(y, z) = \cos y \Rightarrow g(y, z) = \sin y + h(z)$, and we have now $f = 2x^2e^z + \sin y + h(z)$. Now take partial with respect to $z$: $f_z = 2x^2e^z + h'(z)$, and this implies that $h'(z) = 0 \Rightarrow h(z) = K$, a constant. Thus $f = 2x^2e^z + \sin y + K$.

**Green’s Theorem**

Relates line integral of a vector field around a *closed* curve/path $C$ to the double integral over the plane region $D$ bounded by $C$: $\int_{C} \vec{F} \cdot d\vec{r} = \iint_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \iint_{D} \nabla \times \vec{F} \cdot \vec{k} \, dA$. This is another FTC form: integral of a plane region equals integral on its boundary. Note that $\vec{F} \cdot d\vec{r}$ is the tangential component of the vector field to the path. Green’s Theorem can be expressed using the normal component $\int_{C} \vec{F} \cdot \vec{n} \, ds = \iint_{D} \text{div} \vec{F} \, dA = \iint_{D} \nabla \times \vec{F} \, dA$ with $\vec{n}$ being the unit vector normal to the path.

**Surface Area** $\iint_{S} dS$

When the surface is described by $s(x, y, z) = z - g(x, y) = 0$, then $S_A = \iint_{D} |\nabla s| \, dA$. When the surface is defined by parametric equations $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, then $S_A = \iint_{D} |\vec{r}_u \times \vec{r}_v| \, dA$, where
\[ \vec{r}_u = \left\{ \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\}, \] and \[ \vec{r}_v = \left\{ \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\}. \]

Note that these two integrals, for surfaces defined \( z = g(x,y) \), or parameterized \( \vec{r}(x,y) = (x,y,g(x,y)) \), become the familiar integral

\[
S_A = \iint_{D} \sqrt{1 + \left( \frac{\partial g}{\partial x} \right)^2 + \left( \frac{\partial g}{\partial y} \right)^2} \, dA.
\]

**Surface Integrals of Scalar Functions** \[ \iint_{S} f(x,y,z) \, dS \]

With \( s(x,y,z) = z - g(x,y) = 0 \), parameterized: \( \vec{r}(x,y) = (x,y,g(x,y)) \), then the integral becomes

\[
= \iint_{D} f(\vec{r}) \left| \nabla s \right| dA = \iint_{D} f(\vec{r}) \left| \vec{r}_x \times \vec{r}_y \right| dA = \iint_{D} f(x,y,g(x,y)) \sqrt{1 + \left( \frac{\partial g}{\partial x} \right)^2 + \left( \frac{\partial g}{\partial y} \right)^2} \, dA. \]

These integrals can be interpreted as the mass of a lamina/thin sheet with \( S \) shape and mass density \( f \).

**Surface Integrals of Vector Fields** \[ \iint_{S} \vec{F} \cdot d\vec{S} \]

With positive \( \vec{n} \) being the unit vector normal out from the surface \( S \):

\[
\iint_{S} \vec{F} \cdot d\vec{S} = \iint_{S} \vec{F} \cdot \vec{n} \, dS \]

measures the rate of flow through \( S \), also called the flux. The gradient of a surface is a vector normal to the surface, and when the surface \( S \) is given by \( s(x,y,z) = z - g(x,y) = 0 \), then the integral becomes

\[
\iint_{S} \vec{F} \cdot \vec{n} \, dS = \iint_{D} \vec{F} \cdot \left| \nabla s \right| dA = \iint_{D} \vec{F} \cdot \nabla s \, dA. \]

For parameterized surfaces \( \vec{r}(x,y) = (x,y,g(x,y)) \), we have

\[
\iint_{S} \vec{F} \cdot d\vec{S} = \iint_{D} \vec{F} \cdot \vec{r}_x \times \vec{r}_y \, dA
\]

**Divergence Theorem**

Relates integral over a surface \( S \) to the triple integral of the enclosed body \( E \). It is given by

\[
\iint_{S} \vec{F} \cdot d\vec{S} = \iiint_{E} \nabla \cdot \vec{F} \, dV
\]

with the differential surface \( d\vec{S} \) defined above (\( d\vec{S} = \vec{n} \, dS \), which is equal to either \( \nabla s \, dA \), or \( \vec{r}_x \times \vec{r}_y \, dA \)).

This theorem is useful for solving flux problems. This is another FTC form: the triple integral of a vector field for the enclosed body \( E \) is equal to the body surface integral. Most of the time the triple volume integral is the easiest way to calculate surface integrals.

**Stoke’s Theorem**

Relates integral over a surface \( S \) to a line integral around its boundary curve \( C \) counterclockwise; note the similarity with Green’s Theorem which is for planes while Stoke’s Theorem is for any surface. It is given by

\[
\oint_{C} \vec{F} \cdot d\vec{r} = \iint_{S} \nabla \times \vec{F} \cdot d\vec{S} = \iiint_{E} (\nabla \times \vec{F}) \cdot dV, \quad \text{with} \quad d\vec{S} \quad \text{being the differential surface defined above}
\]

(\( d\vec{S} = \vec{n} \, dS \), which is equal to either \( \nabla s \, dA \), or \( \vec{r}_x \times \vec{r}_y \, dA \)).

This theorem is useful for solving flux problems. This is another FTC form: the integral of normal component of vector field over a surface \( S \) is equal to integral around surface boundary curve \( C \). Sometimes the Surface integral is easier, and sometimes the line integral is easier. (NOTE: if the surface is closed, we can use the Divergence Theorem and then

\[
\iint_{S} \text{curl} \vec{F} \cdot d\vec{S} = \iiint_{E} \nabla \cdot (\nabla \times \vec{F}) \, dV = \iiint_{E} \nabla \cdot \vec{F} \, dV = 0
\]