1. Let \( x_n \) be a sequence of real numbers satisfying
\[
|x_{n+1} - x_n| \leq \alpha^n \quad \text{for all } n \geq 1,
\]
where \( \alpha > 1 \) is a constant. Prove \( \lim_{n \to \infty} x_n \) exists.

2. (a) Assume \( \sum_{n=1}^{\infty} a_n \) is convergent and \( \{b_n\} \) is a bounded sequence. Does \( \sum_{n=1}^{\infty} a_nb_n \) converge? Prove or provide a counterexample.

(b) Assume \( \sum_{n=1}^{\infty} |a_n| \) is convergent and \( \{b_n\} \) is a bounded sequence. Does \( \sum_{n=1}^{\infty} a_nb_n \) converge? Prove or provide a counterexample.

3. Consider the series
\[
\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n+x}.
\]
(a) Prove this series is convergent pointwise for all \( x \in [0, 1] \).

(b) Prove this series is uniformly convergent for all \( x \in [0, \varepsilon] \) where \( \varepsilon \) is a constant and \( 0 < \varepsilon < 1 \).

(c) Is this series uniformly convergent for all \( x \in [0, 1] \)? Prove your answer.

4. (a) Define what it means for a bounded function on \([a, b]\) to be Riemann integrable.

(b) Use your definition from (a) to prove the following function \( f(x) \) is Riemann integrable on \([0, 2] \):
\[
f(x) = \begin{cases} 
  x, & 0 \leq x \leq 1 \\
  2x + 1, & 1 < x \leq 2
\end{cases}
\]

5. Let \( K \) be the following set:
\[
K = \left\{ \frac{1}{n} : n \geq 1 \right\} \cup \left\{ -\frac{1}{n} : n \geq 1 \right\} \cup \left\{ \frac{1}{n} - \frac{1}{m} : m, n \geq 1 \right\}
\]
(a) Find all limit points of \( K \).

(b) Prove \( K \) is a compact set using the open cover definition.