Real Analysis Qualifying Exam

Date: June 04, 2011

Duration: 2 Hours

Instructions: This exam consists of 5 questions. Each question is worth 5 points giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. Definitions: A real number $x$ is said to be algebraic if it is a root of a non-zero polynomial with rational coefficients. That is, $x \in \mathbb{R}$ is algebraic if there are numbers $a_0, a_1, \ldots, a_n \in \mathbb{Q}$, not all zero, such that
   \[ a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0. \]
   A real number $x$ is said to be transcendental if it is not algebraic.

   (a) Prove that the set $A$ of all real algebraic numbers is countable.
   
   **Hint:** For $n \in \mathbb{N}$, let $\mathbb{Q}_n[x] = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_0, a_1, \ldots, a_n \in \mathbb{Q}, a_n \neq 0\}$ denote the set of all $n^{th}$ degree polynomials with rational coefficients. For $p \in \mathbb{Q}_n[x]$, let $Z(p) = \{x \in \mathbb{R} \mid p(x) = 0\}$. Then write
   \[ A = \bigcup_{n \in \mathbb{N}} A_n \quad \text{where} \quad A_n = \bigcup_{p \in \mathbb{Q}_n[x]} Z(p). \]

   (b) Use part (a) to prove that the set $\mathcal{T}$ of all real transcendental numbers is uncountable.

2. Let $\{a_n\} \subset \mathbb{R}$ be a sequence of positive real numbers such that
   \[ n \left(1 - \frac{a_{n+1}}{a_n}\right) \geq 2 \quad \text{for all} \quad n \geq N_0 \quad \text{for some} \quad N_0 > 2. \]
   Prove that the series $\sum_{n=1}^{\infty} a_n$ converges.

3. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by
   \[ f(x) = \begin{cases} 
   1 & \text{if } x = 0 \\
   0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\
   \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} 
   \end{cases} \]
   where $m, n \in \mathbb{Z}$ are positive integers with no common divisors.

   (a) Prove that $f$ is discontinuous on $\mathbb{Q} \cap [0, 1]$.

   (b) Prove that $f$ is continuous on $(\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$.

   **Hint:** For part (b) show that for each $n \in \mathbb{N}$, the set $f^{-1}\left(\frac{1}{n}\right)$ is finite.

4. Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a fixed point if $f(x) = x$ for some number $x \in \mathbb{R}$.

   Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $|f'(t)| \leq \alpha$ for all $t \in \mathbb{R}$ and some number $0 < \alpha < 1$. Let $x_1 \in \mathbb{R}$ be arbitrary and define
   \[ x_{n+1} = f(x_n) \quad \text{for} \quad n \in \mathbb{N}. \]
   Prove that $f$ has fixed point $x$ where $x = \lim_{n \to \infty} x_n$.

5. (a) State the definition for a real-valued function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a, b]$.

   (b) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. Let $g : [a, b] \to \mathbb{R}$ be a function such that the set
   \[ E = \{x \in [a, b] \mid g(x) \neq f(x)\} \]
   is finite. Use the definition of Riemann integrability to show that $g$ is Riemann integrable on $[a, b]$ and that
   \[ \int_{a}^{b} g(x) \, dx = \int_{a}^{b} f(x) \, dx. \]

   **Hint:** Consider the function $h(x) : [a, b] \to \mathbb{R}$ defined by $h(x) = f(x) - g(x)$.

   **Note:** If you choose to work with a definition of Riemann integrability different than that stated in part (a), please provide this alternate definition.