Real Analysis Qualifying Exam
Date: September 14, 2011
Duration: 2 Hours

Instructions: This exam consists of 5 questions. Each question is worth 5 points giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. Let \( A \subseteq \mathbb{R} \) be a nonempty subset of real numbers.
   (a) State the definition for a point \( p \in \mathbb{R} \) to be a limit point of \( A \).
   (b) Let \( A' \) denote the set of all limit points of \( A \). Prove that \( A' \) is closed.
   (c) Give an example to show that \( A \) and \( A' \) do not necessarily have the same limit points.

2. Let \( f \) be a differentiable function on the open interval \((a, b)\) and suppose that \( f'(x) \neq 1 \) for all \( x \in (a, b) \). Show that the equation \( f(x_0) = x_0 \) has at most one solution \( x_0 \in (a, b) \).

3. Consider the piecewise defined real-valued function
   \[
   f(x) = \begin{cases} 
   x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
   0 & \text{if } x = 0.
   \end{cases}
   \]
   Prove that \( f \) is differentiable on \( \mathbb{R} \) but that \( f' \) is not continuous at \( x = 0 \). Please completely justify your work for full credit.

4. Show that the series of functions \( \sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} \) converges uniformly in every bounded interval \([a, b] \subset \mathbb{R} \).
   **Hint:** Note that \((-1)^n \frac{x^2 + n}{n^2} = (-1)^n \frac{x^2}{n^2} + (-1)^n \frac{1}{n} \).

5. (a) State the definition for a real-valued function \( f : [a, b] \to \mathbb{R} \) to be Riemann integrable on the interval \([a, b] \).
   (b) Suppose \( f : [a, b] \to \mathbb{R} \) is strictly increasing on \([a, b] \). That is, if \( x < y \) for some \( x, y \in [a, b] \), then \( f(x) < f(y) \). Use the definition of Riemann integrability to show that \( f \) is Riemann integrable on \([a, b] \).
   **Note:** If you choose to work with a definition of Riemann integrability different than that stated in part (a), please provide this alternate definition.