1. Let $A$ and $B$ be closed, disjoint subsets of real numbers. The distance from $A$ to $B$ is defined by

$$d(A, B) = \inf \{ |a - b| : a \in A, b \in B \}.$$ 

(a) Show that if $A = \{ a \}$ is a singleton set, then $d(A, B) > 0$.

(b) Show that if $A$ is compact, then $d(A, B) > 0$.

2. (a) State the definition of convergence for a series of real numbers $\sum_{k=1}^{\infty} a_k$ in terms of partial sums.

(b) Suppose that a series of real numbers $\sum_{k=1}^{\infty} a_k$ converges where $a_k \geq 0$ for all $k \in \mathbb{N}$. Show that $\sum_{k=1}^{\infty} a_{2k}$ converges.

3. Let $A, B \subset \mathbb{R}$ be subsets of real numbers and let $\alpha \geq 1$. Suppose $f : A \to B$ is a surjective, real-valued, continuous function with $|f(x) - f(y)| \geq |x - y|^\alpha$ for all $x, y \in A$.

(a) Show that $f$ is invertible (i.e. show $f^{-1} : B \to A$ exists).

(b) Show $f^{-1}$ is uniformly continuous on $A$.

4. Let $f \geq 0$ be a non-negative, real-valued, continuous function on $[0, 1]$. Show that

$$g(x) = \sum_{k=1}^{\infty} \left( \frac{f(x)}{1 + f(x)} \right)^k$$

is well-defined and continuous on $[0, 1]$.

5. (a) State the definition for a real-valued function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a, b]$.

(b) Suppose $f : [0, 1] \to \mathbb{R}$ is a bounded real-valued function with the property that $f$ is Riemann integrable on

$$\left[ \frac{1}{n}, 1 - \frac{1}{n} \right]$$

for all $n \geq 2$.

Use the definition of Riemann integrability to show that $f$ is Riemann integrable on $[0, 1]$.

Note: If you choose to use a definition of Riemann integrability different than that stated in part (a), please provide this alternate definition.