1. Let $A \subseteq \mathbb{R}$ be a non-empty bounded subset of $\mathbb{R}$.

(a) Prove that there exists a sequence of real numbers in $A$ that converge to $\sup A$.

(b) Let $b \in \mathbb{R}$. Suppose that $b$ is an upper bound for $A$ and that there exists a sequence of real numbers in $A$ that converge to $b$. Prove that $b = \sup A$.

2. (a) Let $\{a_k\}$ be a sequence of real numbers that converge to 0. Prove that $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\sum_{k=1}^{\infty} (a_{2k-1} + a_{2k})$ is convergent.

(b) Provide an example of a sequence of real numbers $\{a_k\}$ such that the series $\sum_{k=1}^{\infty} (a_{2k-1} + a_{2k})$ is convergent but the series $\sum_{k=1}^{\infty} a_k$ is divergent. Please justify your claim for full credit.

3. Let $n \in \mathbb{N}$ and let $c_0, \ldots, c_n \in \mathbb{R}$. Let $a \in \mathbb{R}$ and suppose that

$$c_0 + \frac{c_1}{2} + \cdots + \frac{c_{n-1}}{n} + \frac{c_n}{n+1} = a.$$

Show that there exists a real number $x \in \mathbb{R}$ such that

$$c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + c_n x^n = a.$$

**Hint:** Construct a function that is motivated by the given information.

4. Let $\{a_n\}$ be a sequence of real numbers that converge to $a \in \mathbb{R}$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a uniformly continuous function. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f_n(x) = f(x + a_n)$ for $x \in \mathbb{R}$. Show that the sequence of functions $\{f_n\}$ converge uniformly to $g$ on $\mathbb{R}$ where $g(x) = f(x + a)$.

5. For each $n \in \mathbb{N}$, let $x_n = \frac{1}{n}$ and define $f_n : [0,1] \rightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \{x_1, \ldots, x_n\} \\ 1 & \text{if } x \in [0,1] \setminus \{x_1, \ldots, x_n\} \end{cases}.$$

(a) Using the definition of Riemann integrability, show that for each $n \in \mathbb{N}$, the function $f_n$ is Riemann integrable on $[0,1]$ with $\int_0^1 f_n(x) \, dx = 1$.

(b) Determine the pointwise limit of the sequence of functions $\{f_n\}$ on $[0,1]$ and prove that this limit is Riemann integrable on $[0,1]$. 