Do all five problems.

1. Let $G$ be a group and let $\text{Aut}(G)$ denote the group of automorphisms of $G$. Let

$$\text{Inn}(G) = \{ \phi \in \text{Aut}(G) \mid \exists h \in G \text{ such that } \phi(g) = h^{-1}gh \ \forall g \in G \}$$

be the subgroup of inner automorphisms. Show that $\text{Inn}(G) \triangleleft \text{Aut}(G)$.

2. Prove the Third Isomorphism Theorem: For any group $G$ with normal subgroups $H \triangleleft G$ and $K \triangleleft G$, where $H \subseteq K$ we have:

(a) $K/H \triangleleft G/H$

(b) $(G/H)/(K/H) \cong G/K$

3. Let $R, S$ denote commutative rings with unities. Let $\phi : R \to S$ be a homomorphism of rings. Prove that if $J \subset S$ is a prime ideal then $\phi^{-1}(J)$ is either equal to $R$ or a prime ideal of $R$.

4. Let $V$ be the vector space of upper triangular $2 \times 2$ matrices over $\mathbb{R}$. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Show that $(A, B, C)$ is a basis of $V$.

(b) Define an inner product on $V$ by $\langle X, Y \rangle = \text{tr}(XY^T)$. Find the orthogonal projection of $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ onto the subspace spanned by $A$ and $B$. (Here $\text{tr}(M)$ denotes the trace of the matrix $M$.)

5. Let $T : V \to V$ be a linear map on an $n$-dimensional vector space $V$.

(a) Suppose $v_1, v_2, \ldots, v_n$ are non-zero eigenvectors of $T$, associated with distinct eigenvalues. Show that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent. Conclude that this forms a basis.

(b) Suppose each $v_i$ is also an eigenvector of $S : V \to V$ (with not necessarily the same eigenvalues). Show that $ST = TS$. 