Do all five problems.

1. Let $\phi : G \to H$ be a group homomorphism. Suppose that $|G| = 18$ and $|H| = 15$ and that $\phi$ is not trivial. What is $|\ker(\phi)|$?

2. Prove that if $G$ is a group for which $G/Z(G)$ is cyclic then $G$ is abelian. [Here $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$ is the center of $G$.]

3. Let $S$ and $T$ be linear operators such that $ST = TS$. Let $W$ be an eigenspace of $T$. Show that $W$ is invariant under $S$.

4. Let $R$ be a commutative ring with 1 and let $N = \{r \in R \mid r^n = 0 \text{ for some } n\}$ be the nilradical of $R$ (i.e. the set of all nilpotent elements). You may assume $N$ is an ideal. Prove the following are equivalent:
   
   (a) Every element of $R$ is either nilpotent or a unit.
   
   (b) $R/N$ is a field.

5. Let $V$ be an inner product space (with a positive-definite inner product). Prove that any finite set of non-zero vectors whose elements are mutually orthogonal is linearly independent.