Do all five problems.

1. Let $G$ be any group.
   (a) Let $\phi : G \to G$ be defined by $\phi(g) = g^2$ for all $g \in G$. Prove that $\phi$ is a homomorphism if and only if $G$ is abelian.
   (b) If $G$ is abelian and finite, show that $\phi$ is an automorphism if and only if $G$ has odd order.

2. Let $G$ be a group and let $S = \{xyx^{-1}y^{-1} | x, y \in G\}$. Prove: If $H$ is a subgroup of $G$ and $S \subseteq H$, then $H$ is a normal subgroup of $G$.

3. Prove that in an integral domain $D$ every prime element is an irreducible.

4. Find necessary and sufficient conditions on $\alpha, \beta, \gamma \in \mathbb{R}$ such that the matrix
   \[
   \begin{pmatrix}
   1 & \alpha & \beta \\
   0 & 0 & \gamma \\
   0 & 0 & 1
   \end{pmatrix}
   \]
   is diagonalizable over $\mathbb{R}$.

5. Let $\mathcal{M}_n(\mathbb{R})$ be the vector space of $n \times n$ matrices with entries in $\mathbb{R}$ and let $\mathcal{S}$ and $\mathcal{Z}$ denote the set of real $n \times n$ symmetric and skew-symmetric matrices, respectively (recall that an $n \times n$ matrix $A$ is skew-symmetric if $A^T = -A$).
   (a) Show that $\dim(\mathcal{S}) = \frac{n(n+1)}{2}$. A brief justification is sufficient.
   (b) Let $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ be the linear transformation defined by
   \[
   T(A) = \frac{A + A^T}{2}
   \]
   for all $A \in \mathcal{M}_n(\mathbb{R})$. Prove that $\ker(T) = \mathcal{Z}$ and $\text{im}(T) = \mathcal{S}$.
   (c) Compute $\dim(\mathcal{Z})$. 