Consumer Rationing and the Cournot Outcome

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Abstract

For a symmetric two-stage game, where firms first choose capacities, then compete in prices, Kreps and Scheinkman (Bell Journal of Economics, 1983, 14(2), pp. 326-337) prove that under efficient rationing the Nash equilibrium coincides with the Cournot equilibrium. We extend the model to include asymmetric costs and provide new results showing that the capacity choice game is dominance-solvable, just like the Cournot game. Further, we provide a simple sufficient condition, under which the dominance-solvable result extends to proportional rationing. The results provide new insights into the robustness of Cournot coincidence under alternate demand rationing schemes.

KEYWORDS: Bertrand-Edgeworth, demand rationing, Cournot, judo economics, capacity pre-commitment

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Introduction

For a two-stage game, where firms first commit to capacities, then compete in prices, Kreps and Scheinkman (1983) (henceforth K&S) prove that the unique Nash equilibrium outcome coincides with the Cournot equilibrium outcome. This result offers an attractive dynamic interpretation to the Cournot model. Two restrictive assumptions, firm symmetry and efficient rationing in the pricing subgames, are fundamental to K&S’s proof of their theorem. In an influential paper, Davidson and Deneckere (1986) illustrate that the Kreps and Scheinkman result does not always hold under proportional rationing. But they do not specify conditions under which the K&S result will hold under such rationing. Our goal is to establish new results regarding the Cournot model as a reduced-form for the two-stage game — regardless of firms’ cost structure or the specific rationing rule.

Starting from the K&S two-stage game, define the capacity choice game as the simultaneous choice game in capacities whose payoff function is derived by rolling back the Nash equilibria of the pricing subgames. Thus, a pair of capacities is a Nash equilibrium of the capacity choice game if, and only if, it is part of a subgame perfect equilibrium of the two-stage game.

We provide new results regarding capacity choice games with efficient and proportional rationing. With efficient rationing, we prove the capacity choice game is dominance-solvable, just like the Cournot game. Thus the two-stage and the Cournot games are even more closely linked than one might suspect from K&S.\(^1\) We also prove that under a simple condition, the capacity choice game with proportional rationing is dominance-solvable. Indeed, the condition permits us to extend the result to a broad class of rationing schemes between the efficient and proportional rules. These results are based on intuition from judo economics and application of results from the literature on games with strategic complementarities.\(^2\)

All our results are built upon an instructive unifying intuition derived from Gelman and Salop (1983) (henceforth G&S). G&S show that a monopolist

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\(^1\)Gabay and Moulin (1980) introduce the concept of dominance-solvability. A game is *dominance-solvable* if a sequence of strictly dominated strategies can be iteratively deleted until each player has a unique remaining strategy. Moulin (1984) showed conditions under which the Cournot game is dominance-solvable.

\(^2\)The literature on monotone comparative statics and supermodular games provides the basis for analyzing games with strategic complementarities. Much of the theory was developed in Topkis (1978, 1979), Vives (1990), and Milgrom and Roberts (1990) and continues to be expanded. Vives (2005, 2008) provide detailed overviews of this literature.
of unlimited size will be unable to deter a potential entrant if the entrant can credibly restrict its capacity. As a result, the entrant will choose to be “small” and price sufficiently low that the monopolist will not find it profitable to undercut. This strategy is what G&S call “judo economics” because the incumbent’s large size is being used against it.

To apply judo economics to the K&S two-stage game we extend the G&S analysis by permitting the incumbent to choose his capacity, taking into account how vulnerable he will be to his rival’s judo given any possible capacity choice. (By contrast, G&S restrict their analysis to the case in which the incumbent has fixed, unlimited capacity.) This extension leads to important new judo questions: Will the incumbent choose to be sufficiently large so that he will be vulnerable to judo? How small must he be to avoid being vulnerable? Finally, focusing on the entrant rather than the incumbent: How big will an entrant choose to be, given his rival is vulnerable to judo?

Until Section 4 of the paper, we will follow K&S by assuming efficient rationing. We prove that under efficient rationing the following three Judo Principles always hold: 1) a firm will never choose to be the victim of judo, 2) a firm will be the victim of judo if and only if it chooses a capacity larger than a specified threshold level, and 3) a judo player will not be too big or too small. Notice these three Judo Principles provide answers to the above three judo questions.

The establishment of the three Judo Principles allows us to prove that the capacity choice game is dominance-solvable, just like the Cournot game. The Judo Principles imply that the best response of each firm is a monotonic nonincreasing correspondence. Since the game involves two firms, we can re-frame the game as one with monotonic nonincreasing best responses and apply results from the literature for games with strategic complementarities. Particularly, we apply a result from Milgrom and Shannon (1994), which states that a game with monotonic nonincreasing best responses and a unique pure strategy Nash equilibrium is dominance-solvable. A dominance-solvable game has no mixed strategy equilibria; hence the unique equilibrium of the capacity

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3In our context, we call a firm who uses a judo strategy a judo player, that is, a firm who chooses to be smaller than its rival when the Nash equilibrium of the ensuing pricing subgame will involve mixed strategies.

4In G&S’s model, firms price sequentially, with the (inevitably) smaller entrant choosing its price first. Deneckere and Kovenock (1992) prove that the capacity constrained sequential pricing game with the smaller firm pricing first yields the same expected revenue as the simultaneous pricing game in K&S’s model.
choice game coincides with the unique equilibrium of the Cournot game.

The penultimate section extends our analysis to proportional rationing (and indeed to a broad class of rationing rules between the efficient and proportional rules). We first prove that, if the sum of the Cournot capacities is less than a zero cost monopolist’s capacity, then the Cournot outcome is an equilibrium outcome of the two-stage game under proportional rationing. Uniqueness results are based on extensions of the Judo Principles to proportional rationing. If each firm’s marginal cost at its Cournot capacity is greater than the zero cost monopoly price, then the Judo Principles are true with proportional rationing.\footnote{By contrast, under efficient rationing, the three Judo Principles \textit{always} hold.}

The proofs of the best response properties are constructed indirectly, by way of an extension of the judo economics methodology. In order to deal with the intractability of the expected revenue under proportional rationing, we prove the K&S result for a game with sequential judo pricing and then show this implies the result for the simultaneous pricing game. Based on the principles, the firms’ best response correspondences are monotonic nondecreasing. Thus, we are able to once again use the result of Milgrom and Shannon (1994) to prove that the capacity choice game is dominance-solvable for the Cournot capacities.

We also present examples showing when the above condition holds. To illustrate, if demand and costs are linear, then the condition will be satisfied for 1/4 of all admissible cost specifications and 1/2 of all symmetric specifications.

Our analysis of the capacity choice game is built upon established results regarding capacity constrained pricing subgames. In terms of efficient rationing, our analysis is highly dependent on results of K&S as well as the literature that has followed. Expanding on K&S, Osborne and Pitchik (1986) characterize the equilibria of the efficient rationing pricing subgame under more general assumptions. Deneckere and Kovenock (1992) show the relationship between the expected revenue of sequential and simultaneous pricing subgames; results which are heavily utilized in our analysis. In terms of proportional rationing, Davidson and Deneckere (1986) and Allen and Hellwig (1991) provide characterizations of the pricing subgame under proportional rationing which provide our basis for analysis of the capacity choice game.

There are also closely related contributions focusing on equilibria of different capacity choice games. Gruyer (2009) extends the K&S result to asymmetric costs of capacity when firms are restricted to choose pure strategy capacities. Boccard and Wauthy (2000, 2004) extend the K&S result to oligopoly with
costly adjustment of capacity in the second stage. Deneckere and Kovenock (1996) extend the analysis with efficient rationing to include asymmetric costs of production up to capacity. It is shown that with sufficient asymmetry in these costs the K&S result will no longer hold. Going beyond G&S, Allen (1993) analyzes the equilibrium of a sequential capacity game with simultaneous pricing. Allen et al. (2000) generalize the game with sequential capacity choices and build upon Deneckere and Kovenock (1996) allowing firms to have asymmetric costs up to capacity. Depending on cost specifications, they find a rich variety of equilibria. This includes cases of a stochastic judo-like outcome, the Stackelberg outcome, or a reverse judo tactic used by the incumbent when it is cost disadvantaged.

The paper is organized as follows. In Section 2, we lay out the basic assumptions of the model, characterize the Cournot equilibrium, characterize the pricing subgames of the two-stage game, and introduce some concepts for analyzing the rolled-back capacity choice game. Section 3 contains all the main results for the capacity choice game with efficient rationing. Section 4 extends the analysis to the capacity choice game with proportional rationing. It also includes some examples. The concluding section hints at further extensions of our results.

2 The Model Basics

Consider an industry with two firms producing a single homogeneous product. The two firms compete in a game where they first choose capacities independently and simultaneously; then these choices are made public, and prices are chosen independently and simultaneously. The market demand function is $D : \mathbb{R}^+ \mapsto \mathbb{R}^+$. The inverse demand is $P : \mathbb{R}^+ \mapsto \mathbb{R}^+$. Each firm’s capacity provides an absolute limit on the number of units it can produce: its cost of production is zero up to capacity, and infinite for any quantity beyond. The two firms are identified by their capacity cost functions $(a, b)$, and in the ensuing pricing subgame by their actual capacities $(x, y)$. Demand is bounded when the market price is zero, so $\bar{X} = D(0) < \infty$. The following three assumptions are maintained throughout the paper.

**Assumption 1** $P(x)$ is twice-continuously differentiable, positive, strictly decreasing, and concave for all $x \in (0, \bar{X})$. In addition, $P(x) = 0$ for all $x \geq \bar{X}$.

**Assumption 2** The firms’ capacity cost functions are $a : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and
Both functions are twice-continuously differentiable, increasing, convex, and satisfy $a(0) = b(0) = 0$.

**Assumption 3** Each firm’s cost of capacity permits positive profit: $a'(0) < P(0)$ and $b'(0) < P(0)$.

It is well known that under the above assumptions there exists a unique maximum to $pD(p)$, we label this $p^m$. We will refer to this price as the zero cost monopoly price, or just monopoly price. Further we denote by $x^m = D(p^m)$, the monopoly quantity and $\pi^m = p^m \cdot x^m$, the monopoly revenue.

### 2.1 The Cournot Game

We first present the basic features of the Cournot game with asymmetric costs. We typically take the generic firm to have cost function $a$ and to choose capacity $x$, while its rival has cost function $b$ and chooses $y$. Denote the generic firm’s Cournot profit function by

\[
\Pi_a^c(x, y) = P(x + y)x - a(x).
\]

This function is defined for all $(x, y) \in \mathbb{R}^2_+$; it is twice continuously differentiable and strictly concave for all $x \in (0, X - y)$. The generic firm’s *Cournot best response* function to its rival’s capacity is

\[
\hat{r}_a(y) = \arg \max_{x \geq 0} \Pi_a^c(x, y).
\]

The Cournot best response function for a firm with zero cost is

\[
r(y) = \arg \max_{x \geq 0} P(x + y)x.
\]

The following preliminary result is from K&S.

**Fact 1 (Kreps and Scheinkman)** (a) For every $a$ as above, $r_a$ is nonincreasing in $y$ and $r_a$ is continuously differentiable and strictly decreasing over the range where it is strictly positive. (b) $r'_a \geq -1$ with strict inequality for $y$ such that $r_a(y) > 0$, so $y + r_a(y)$ is non-decreasing in $y$. (c) If $a$ and $b$ are two cost functions such that $b > a$, then $r_b < r_a$. (d) If $y > r_a(y)$, then $r_a(r_a(y)) < y$. 

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It is well known that under Assumptions 1-3 there exists a unique Cournot equilibrium, \((x^*(a, b), y^*(a, b))\); indeed, under these assumptions the Cournot game is dominance-solvable (Moulin (1984)). Throughout the rest of the paper we will drop the cost arguments and simply write the Cournot equilibrium as \((x^*, y^*)\).  

3 Efficient Rationing

Now let us begin to analyze the two-stage game with efficient rationing.

3.1 The Pricing Subgames

The character of the Nash equilibrium in any pricing subgame will depend on firms’ previous capacity choices, \((x, y)\).

Following K&S, we will assume until Section 4 that demand is rationed according to the surplus maximizing or efficient rationing rule. Dasgupta and Maskin (1986b) prove that, given any capacity choices, the ensuing pricing subgame will have a mixed strategy Nash equilibrium. Our characterization of this equilibrium will be based on results established in K&S and Deneckere and Kovenock (1992).

While any given pricing subgame has a unique Nash equilibrium, there are four species of equilibrium pricing, depending on firms’ ex ante capacity choices \((x, y)\). Accordingly, define the four regions:

\[
\begin{align*}
C &= \{ (x, y) \in \mathbb{R}^2_+ \mid x \leq r(y) \text{ and } y \leq r(x) \}, \\
M^L_a &= \{ (x, y) \in \mathbb{R}^2_+ \mid (x, y) \notin C, y < \bar{X} \text{ and } x \geq y \}, \\
M^S_a &= \{ (x, y) \in \mathbb{R}^2_+ \mid (x, y) \notin C, x < \bar{X} \text{ and } x < y \}, \\
B &= \{ (x, y) \in \mathbb{R}^2_+ \mid \min\{x, y\} \geq \bar{X} \}.
\end{align*}
\]

The Nash equilibrium expected revenue for firm \(a\) in each region is given by the function

\[
R^e(x, y) = \begin{cases} 
P(x + y)x & \forall (x, y) \in C \\
\min\{P(r(y) + y)r(y), p^J(x, y)x\} & \forall (x, y) \in M^L_a \\
p^J(x, y)x & \forall (x, y) \in M^S_a \\
0 & \forall (x, y) \in B,
\end{cases}
\]

\textsuperscript{6} If capacity costs are asymmetric enough, the Cournot equilibrium will involve only one firm with positive capacity.
where the superscript “e” stands for “efficient rationing.” The correspondence $p^J(x, y)$ is the lower bound of the support of the firms mixed strategies (it will be formally defined shortly).

![Figure 1: Equilibrium pricing regions by capacity pair](image)

The four pricing regions are illustrated in Figure 1. In region $C$, the “Cournot region,” each firm’s capacity is smaller than its zero cost Cournot best response to its rival’s capacity; hence pricing will simply clear the market à la Cournot competition. In region $B$, the “Bertrand region,” each firm has enough capacity to cover the entire market at a price of zero; hence pricing will be at marginal cost (zero) à la Bertrand competition. Thus, in region $C$ competition will be soft, while in region $B$ it will be cutthroat. Capacities between these two extremes lead to mixed strategy pricing.

We label $p^J(x, y)$ the judo safe price based on the fact that it is the equilibrium price charged by the smaller firm in the sequential pricing game where the smaller firm prices first. This is similar to a reduced judo game where both firms capacities are primitives. While in G&S’s formulation, the second movers capacity is fixed (and arbitrarily large), but the first mover is free to choose its capacity. This interesting coincidence is formally established as Theorem 3 in Deneckere and Kovenock (1992).

**Fact 2 (Deneckere and Kovenock)** For capacities $(x, y) \in M^S_a$, the equi-
librium of the sequential pricing game with firm a pricing first, is such that \( p_a = p'(x, y) \) and \( p_b = P(r(x) + x) \). The firms’ equilibrium revenues in this sequential game are the same as equilibrium expected revenues in the simultaneous price game with capacities \((x, y)\).

The judo safe price takes two possible forms which we label \( p^\infty(x) \) and \( p(x, y) \).

We denote by \( p^\infty(x) \), the profit maximizing price such that firm a is safe from being undercut when its rival has essentially unlimited capacity. (The superscript “\( \infty \)” indicates the rival’s capacity is essentially infinite.) Mathematically, this is given by the function

\[
(5) \quad p^\infty(x) = \min \left\{ p \in \mathbb{R}_+ \mid pD(p) = P(r(x) + x)r(x) \right\},
\]

which is defined for all \( x \geq 0 \), and is twice continuously differentiable for all \( x \in (0, \bar{X}) \).

Let \( y(x) = D(p^\infty(x)) \), and note that \( y(x) \geq y(0) \) for all \( x \geq 0 \). Firm b has “essentially unlimited capacity” means \( y > y(x) \) because, even if it undercuts firm a, the larger firm has more than enough capacity to satisfy the whole market.

Let us now suppose that firm b has capacity \( y < y(x) \). We denote by \( p(x, y) \), the lower bound of the price when \( y < y(x) \). Formally, \( p(x, y) \) is defined implicitly by

\[
(6) \quad p(x, y)y = P(r(x) + x)r(x).
\]

As a function, \( p(x, y) \) is defined for all \( y > 0 \) and \( x \geq 0 \), and is twice continuously differentiable for all \( x \in (0, \bar{X}) \).

The judo safe price is the maximum of the above two prices:

\[
(7) \quad p'(x, y) = \max \{p(x, y), p^\infty(x)\} = \begin{cases} 
p(x, y) & \text{if } y \in (0, y(x)] 
p^\infty(x) & \text{if } y \geq y(x).
\end{cases}
\]

### 3.2 The Capacity Choice Game

We begin our study of the capacity choice game with some preliminary definitions. If the firms choose capacities \((x, y) \in M^S_a \) with \( y < y(x) \), the judo safe price will be \( p(x, y) \) and the judo player’s expected profit will be

\[
(8) \quad \Pi_a(x, y) = p(x, y)x - a(x).
\]
Viewed as an abstract function, $\Pi_a(x, y)$ is defined for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$, and is twice-continuously differentiable for all $x \in (0, X)$. For any given $y$, this function has a unique maximizer when firm $a$ is restricted to choosing a capacity $x \leq y$, denoted by

$$x_a(y) = \arg \max_{x \in [0, y]} \Pi_a(x, y). \quad (9)$$

Similarly, if the firms choose capacities $(x, y) \in M_{a}^S$ with $y \geq y(x)$, the judo safe price will be $p^\infty(x)$ and the judo player’s expected profit will be

$$\Pi^\infty_a(x) = p^\infty(x)x - a(x). \quad (10)$$

Viewed as an abstract function, $\Pi^\infty_a(x)$ is defined for all $x \in \mathbb{R}_+$, and is twice continuously differentiable for all $x \in (0, X)$. Since its graph is single peaked, it too has a unique maximizer, which we denote by

$$x_a = \arg \max_{x \geq 0} \Pi^\infty_a(x). \quad (11)$$

The following proposition establishes that the programs in (9) and (11) have unique solutions.

**Proposition 1** Both programs (9) and (11) have unique solutions labeled $x_a(y)$ and $x_a$, respectively.

The proof of all results not included in the main text are provided in the appendix.

The *capacity choice profit function* for firm $a$ is given by

$$\Pi^c_a(x, y) = R^c(x, y) - a(x), \quad (12)$$

and the corresponding *capacity choice best response* of firm $a$ to any capacity $y$ of its rival is given by

$$r^c_a(y) = \arg \max_{x \in [0, X]} \Pi^c_a(x, y). \quad (13)$$

Note that the function $\Pi^c_a(x, y)$ is bounded and continuous for all $(x, y) \in \mathbb{R}^2$. Hence, for any $y \in [0, X]$, it attains a maximum over all $x \in [0, X]$. A continuous function on a compact rectangle in $\mathbb{R}^2$ is uniformly continuous on that rectangle; and, in turn, a uniformly continuous function on a compact
space has a closed set of maximizers. Thus the set \( r_a^e(y) \) is closed for all \( y \in [0, \bar{X}] \).

Anticipating, we will show in due course that the best response correspondence \( r_a^e \) is single valued almost everywhere, indeed a continuous function on \([0, \bar{X}]\) except for possibly one discontinuity point. We also will show that if \( y \) is sufficiently large, \( r_a^e(y) \) will equal either \( x_a(y) \) or \( x_a \), hence the rival will fall victim to judo.

### 3.3 Judo Economics in Action

This section establishes the judo principles and their consequences. The judo principles are properties of the extended judo game, which under efficient rationing is analogous to a single firm’s best response correspondence in the capacity choice game. These principles yield a characterization of the best response correspondence, which is the basis for analysis of the capacity choice game.

The 1st *Judo Principle* is the basic property that no firm will ever choose to be the victim of judo. That is, no firm will ever find it optimal to be (weakly) larger than its rival if the ensuing Nash equilibrium pricing subgame will be in the mixed pricing region. Instead, the firm will prefer to be smaller and avoid being attacked by the judo strategy. The intuition behind this principle is simple. If firm \( a \) is larger and the Nash equilibrium of the pricing subgame involves mixed strategies, the larger firm makes the same expected revenue for all capacities greater than or equal to \( r(y) \). So, because capacity is costly, the larger firm increases its expected profit by reducing its capacity to \( r(y) \) starting from any \( x > r(y) \), which moves it out of the region \( M_a^L \).

The 2nd *Judo Principle* provides the exact circumstances when a firm will choose to use judo: its rival must be larger than a specific level \( \bar{y} \). To determine this level, consider a Cournot game in which firm \( a \) faces a zero cost rival; let \( y \) be the equilibrium capacity of the zero cost firm in this game, \( \bar{y} = r(r_a(y)) \).

This turns out to be the threshold capacity that determines when firm \( a \) will find it optimal to use judo. If the rival builds capacity less than or equal to \( \bar{y} \), then firm \( a \) will respond with a capacity \( x \in r_a^e(y) \) such that \((x, y)\) remains in the Cournot pricing region. But if firm \( b \) builds capacity \( y > \bar{y} \), then firm \( a \) will find it optimal to respond with a capacity \( x \in r_a^e(y) \) such that \((x, y) \in M_a^S \).

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7We do not have to worry about \( y = 0 \) because the expected revenue function \( R^e(x, y) \) equals either \( P(x + y)x \) or \( P(r(y) + y)r(y) \) at \( y = 0 \). Hence, the fact that \( p^f(x, y) \) is not defined for \( y = 0 \) does not affect the function \( \Pi_a^e(x, y) \).
Hence, if and only if \( y > y \) will firm \( a \) find it optimal to use judo against its rival. The threshold capacity \( y \) is illustrated in Figure 2.

The 3\textsuperscript{rd} \textit{Judo Principle} states that a firm will choose to be smaller as a judo player than otherwise: a judo player will never find it optimal to be larger than \( r_a(y) \) when facing a rival of size \( y > y \).

Consider the following extension of the game studied by G&S, which starts with an incumbent having some fixed capacity \( y > y \). (Gelman and Salop focused on the case when the incumbent has unlimited capacity.) An entrant with cost function \( a \) now must decide on both his capacity \( x \) and the per unit price \( p_x \) he will charge for his product. Finally the incumbent, after observing the judo player’s strategy \( (x, p_x) \), sets a unit price \( p \) for his product. The entrant in the G&S game will pick its size to maximize its judo profit, just like firm \( a \) in the capacity game. In particular, it will pick \( x = x_a(y) \) when \( y \in (y, \bar{y}] \), and it will pick \( x = x_a \) when \( y \geq \bar{y} \). In other words, firm \( a \)'s behavior in the capacity game when \( y > y \) can be interpreted as a judo strategy. To round out the story, having chosen the capacity \( x = r_a(y) \), the entrant in the G&S game will set its price at the judo safe price \( p_a = p^J(x, y) \), as already shown in Section 3.1, and the incumbent will respond with the Stackelberg follower’s price \( p_b = P(r(x) + x) \).

The following lemma provides a characterization of an arbitrary firm’s ca-
capacity game’s best response correspondence, which summarizes the three judo principles. As a preliminary, we provide a formal definition of a monotonic nonincreasing (nondecreasing) correspondence.

**Definition 1** Take a set on \( S \subset \mathbb{R} \). A correspondence \( \phi : S \rightarrow S \) is monotonic nonincreasing (nondecreasing), if for all \( s, s' \in S \) such that \( s > s' \), for all \( r \in \phi(s) \) there exists \( r' \in \phi(s') \) such that \( r' \leq (\geq) r \) and for all \( r' \in \phi(s') \) there exists \( r \in \phi(s) \) such that \( r \leq (\geq) r' \).

Also define the function

\[
 r_a(y) = \begin{cases} 
 r_a^{-1}(y) & \text{if } y \in [0, x^m] \\
 0 & \text{if } y \in [x^m, \bar{X}] 
\end{cases}
\]

where \( x^m \) is a firm’s monopoly output if it has zero cost.

**Lemma 1** For all \( y \in [0, \bar{X}] \), the best response correspondence of firm \( a \) is monotonic nonincreasing and described below:

\[
 r_a^e(y) = \begin{cases} 
 r_a(y) & \text{if } y \in [0, y] \\
 x_a(y) & \text{if } y \in [y, \bar{y}] \\
x_a & \text{if } y \in [\bar{y}, \bar{X}] 
\end{cases}
\]

An example of a best response correspondence described by Lemma 1 is illustrated in Figure 3.

Notice the correspondence is actually a continuous function except perhaps at \( y = \bar{y} \); at this point it may have two values. The critical point \( \bar{y} \) is implicitly defined by the equation

\[
 \Pi_a^\infty(x_a) = \Pi_a(x_a(\bar{y}), \bar{y})
\]

(recall Equations 8–11 in Section 3.1). So, when \( y = \bar{y} \), the judo player is indifferent between choosing the capacity \( x_a \) that maximizes \( \Pi_a^\infty(x) \) and the larger capacity \( x_a(\bar{y}) \) that maximizes \( \Pi_a(x, \bar{y}) \). In exceptional examples, \( x_a = x_a(\bar{y}) \), in which case \( r_a^e(y) \) will be a continuous function throughout its domain.

Based on Lemma 1, we show that \((x^*, y^*)\) is the unique Nash equilibrium of the capacity choice game. In fact, we prove a much stronger property: the capacity choice game is dominance-solvable.

**Theorem 1 (dominance-solvability)** \((x^*, y^*)\) is the dominance-solvable solution for the capacity choice game.
The proof is done in three sequential parts. First, we show based on the 1\textsuperscript{st} Judo Principle that with symmetric costs the unique pure strategy Nash equilibrium is \((x^*, y^*)\). Next, we extend this result to asymmetric costs using the 2\textsuperscript{nd} Judo Principle. Finally, we use 3\textsuperscript{rd} Judo Principle to prove the capacity choice game is dominance-solvable. Our argument for dominance-solvability is based on general results pertaining to games with strategic complementarities. Particularly, we show that we can re-frame the capacity choice game so that it satisfies the conditions of Theorem 12 in Milgrom and Shannon (1994).

**Proof of Theorem 1. Part 1 (K&S in pure strategies)** Based on the fact that \((x^*, y^*)\) is a Cournot equilibrium, the only potential profitable deviations would be to \((x, y^*) \in M^L_d\), which we know cannot improve profits from the 1\textsuperscript{st} Judo Principle. Hence, \((x^*, y^*)\) is an equilibrium. To show uniqueness, notice there cannot be an equilibrium with mixed pricing because one firm would have to be in \(M^L_d\); and \((x, y) \in B\) cannot be an equilibrium because \(x^o = 0\) would be a profitable deviation. Therefore, there can only be an equilibrium \((x, y) \in C\). Since \((x^*, y^*)\) is the unique Cournot equilibrium, any other \((x, y) \in C\) must have a profitable deviation in the Cournot game. But any deviation from \((x, y)\) yields at least as much profits in the capacity choice game as in the Cournot game. Therefore, \((x^*, y^*)\) must also be the unique equilibrium of the capacity choice game.
**Part 2** (Generalized K&S in pure strategies) Based on the fact that \( y^* = r_b(r_a(y^*)) < y = r(r_a(y)) \) and \( x^* = r_a(r_b(x^*)) < x = r_a(r_b(x)) \), the capacity choice best responses coincide with the best responses for the Cournot game: \( r_a(y^*) = r_a(y) \) and \( r_b(x^*) = r_b(x) \). Thus \((x^*, y^*)\) is an equilibrium. Uniqueness follows from the argument of Part 1.

**Part 3** (Dominance-solvable) The argument is based on showing that the capacity choice game can be re-framed as a game with strategic complementarities and then appealing to Theorem 12 in Milgrom and Shannon (1994). Adapting Milgrom and Shannon definition of a game with strategic complementarities to our notation we have: for every player \( a \)

1. Player \( a \)'s strategy space is a compact lattice;
2. Player \( a \)'s payoff function is upper semi-continuous in \( x \) for \( y \) fixed, and continuous in \( y \) for fixed \( x \);
3. Player \( a \)'s payoff function is quasisupermodular in \( x \) and satisfies the single crossing property in \((x; y)\).

The first two requirements are satisfied for the capacity choice game, since the interval \([0, \bar{X}] \subset \mathbb{R}\) is a compact lattice and \( \Pi_a(x, y) \) is a continuous function in both arguments. We are left to show that property three is satisfied. To do this, we rewrite our game so that both firms' best response correspondences are monotonic nondecreasing. Based on Lemma 1 and Fact 1, each firms best response correspondence is monotone nonincreasing. Since, our game only has two players we can redefine player \( b \)'s strategy as \( z = -y \) and the best response correspondences of the converted game are trivially monotone nondecreasing.

The connection between monotonic nondecreasing best responses and the third requirement for a game with strategic complementarities is based on the following special case of Theorem 4 in Milgrom and Shannon (1994) modified to our notation.

**Fact 3** (Milgrom and Shannon) Let \( \Pi_a : [0, \bar{X}] \times [0, \bar{X}] \mapsto \mathbb{R} \). Then \( r^e_a(y) \) is monotonic nondecreasing in \( y \) if and only if \( \Pi_a \) is quasisupermodular in \( x \) and satisfies the single crossing property in \((x; y)\).

Now we appeal to a special case of Milgrom and Shannon (1994), Theorem 12 modified to our notation:
Fact 4 (Milgrom and Shannon) In a game with strategic complementarities and a unique pure strategy Nash equilibrium, the Nash equilibrium strategies are the unique serial undominated strategies for players a and b, respectively.

Based on Fact 3, \((x^*, y^*)\) is the unique solution to the capacity choice game by way of iterated deletion of pure strategies that are strictly dominated by other pure strategies. That is, the capacity choice game is dominance-solvable for \((x^*, y^*)\).

The fact that the capacity game has \((x^*, y^*)\) as its unique equilibrium (even including the possibility of mixed-strategy equilibria), follows immediately from Theorem 1.

Corollary 1 \((x^*, y^*)\) is the unique Nash equilibrium of the capacity choice game.

We close this section with the following remark.

Remark 1 K&S’s main theorem is that, when firms have symmetric costs, the Cournot outcome \((x^*, y^*)\) coincides with the unique Nash equilibrium outcome of their two-stage game. Unlike our approach, K&S do not restrict their attention to the subgame perfect equilibria of their game, although their main result immediately implies \((x^*, y^*)\) is the unique subgame perfect Nash equilibrium outcome, which also is our conclusion. One important advantage of the subgame perfect method of analysis, by rolling back the game, is we can illustrate the capacity choice equilibrium using firms’ reaction curves, just as the traditional Cournot equilibrium is illustrated. The Judo Principles (a characterization of capacity best responses) provide the basis to show capacity choice game is dominance-solvable, just like the Cournot game.

In the spirit of K&S we briefly turn our attention to the full set of Nash equilibria for our general two-stage game. This adds the possibility that there may be equilibrium capacity choices \((x^o, y^o) \neq (x^*, y^*)\), sustained by incredible threats off the equilibrium path. Observe that the strongest of all incredible threats are of the following form, here for firm b: If firm a chooses \(x = x^o\), then firm b will price according to the Nash equilibrium of the ensuing pricing subgame. But if firm a chooses any \(x \neq x^o\), then firm b will choose price \(p = 0\). This strategy for firm b leaves firm a with the worst possible residual demand if it deviates from \(x^o\): \(\min\{D(p) - y, 0\}\).

Here we provide a brief heuristic argument for the extension of our uniqueness results to the set of all Nash equilibria of the two-stage game. First notice
we can eliminate equilibria such that \((x, y) \in B\), (since \(x^o = 0\) is always profit increasing) and \((x, y) \in C\) (since the revenue under the incredible threat is the exact same as in the Cournot). Finally, we rule out the possibility of equilibria with mixed strategy pricing. The key insight is to focus on the weakly larger firm: For all \((x, y) \in M^L_a\), a defection to \(x^o = r(y)\) will yield the same expected revenue as \(x\): \(P(y + r(y))r(y)\), but at lower cost. Hence, \(x^o\) is a profit increasing defection.\(^8\)

### 4 Proportional Rationing

In an influential paper, Davidson and Deneckere (1986) challenge the robustness of the Kreps and Scheinkman result. Widening the latter’s analysis to permit any rationing rule between the proportional and efficient rules, they state their goal:

\[\text{... we wish to demonstrate that with any rule in this class, or, in fact, virtually any other one, the Cournot outcome cannot emerge as an equilibrium of the two-stage game.}\]

This statement is fairly easy to misinterpret, so let us first clarify it. It is true that, without efficient rationing, the K&S result does not hold for all cost and demand specifications fitting their assumptions. But it also is true that, even without efficient rationing, the K&S result does hold for some cost and demand specifications fitting their assumptions. The goal of this section is to understand more clearly when the Cournot outcome coincides with that of the two-stage game — regardless of the rationing rule. This will give us a more complete picture of the robustness of the K&S result.

Unless otherwise stated, throughout this section assume proportional rationing. We start by restating the equilibrium characterizing for the pricing subgames with proportional rationing established in Davidson and Deneckere (1986).

\(^8\)It should be apparent that the argument can be easily extended to address mixed strategy capacities.
4.1 The Pricing Subgames

Under the proportional or Beckman rationing rule, the quantity firm \( a \) sells is:

\[
D_a(p_a, p_b) = \begin{cases} 
\min\{x, D(p_a)\} & \text{if } p_a < p_b \\
\min \{x, \max\{D(p_a)/2, D(p_a) - y\}\} & \text{if } p_a = p_b \\
\min \{x, \max\{0, D(p_a)(1 - y/D(p_b))\}\} & \text{if } p_a > p_b.
\end{cases}
\]

As with efficient rationing, the character of the Nash equilibrium in any pricing subgame will depend on the firms’ previous capacity choices \((x, y)\). Again we appeal to Dasgupta and Maskin (1986b) for the existence of a mixed strategy Nash equilibrium in any pricing subgame. Our characterization of this equilibrium is based on results in Davidson and Deneckere (1986).

As under efficient rationing, there are four species of equilibrium pricing, depending on the firms’ ex ante capacity choices \((x, y)\). Accordingly, we define the four regions:

\[
\begin{align*}
\mathbb{C} &= \{(x, y) \in \mathbb{R}_+^2 \mid x + y \leq x^m\}, \\
\mathbb{M}^L_a &= \{(x, y) \in \mathbb{R}_+^2 \mid (x, y) \notin \mathbb{B}, y < \bar{X} \text{ and } x \geq y\}, \\
\mathbb{M}^S_a &= \{(x, y) \in \mathbb{R}_+^2 \mid (x, y) \notin \mathbb{B}, x < \bar{X} \text{ and } x < y\}, \\
\mathbb{B} &= \{(x, y) \in \mathbb{R}_+^2 \mid \min\{x, y\} \geq \bar{X}\}.
\end{align*}
\]

Notice we use blackboard font to identify these regions, to avoid confusion with the corresponding regions under efficient rationing. Compared to the latter, \( \mathbb{B} = B \), but \( \mathbb{C} \subset C \). The four pricing regions under proportional rationing are illustrated in Figure 4.

First let us describe the key features of Nash equilibrium of the simultaneous game as characterized in Davidson and Deneckere (1986). They have shown the existence of a unique mixed strategy equilibrium; we label the equilibrium strategies \((\phi^*_a, \phi^*_b)\). The Nash equilibrium expected revenue for firm \( a \) in each region is given by the function

\[
R^p(x, y) = \begin{cases} 
P(x + y)x & \forall (x, y) \in \mathbb{C} \\
\pi^m \int_{\phi^*_a(x)}^1 (1 - \frac{y}{D(z)}) d\phi^*_b(z) & \forall (x, y) \in \mathbb{M}^L_a \\
\bar{\rho}(x, y)x & \forall (x, y) \in \mathbb{M}^S_a \\
0 & \forall (x, y) \in \mathbb{B},
\end{cases}
\]

where the superscript “\( p \)” stands for “proportional rationing.”

The correspondence \( \bar{\rho}(y, x) \) is the lower bound of the support of both firms’ mixed strategies. This lower bound is composed of two pieces.
Figure 4: Equilibrium pricing by capacity pair under proportional rationing

The first piece \( p^\infty(x) \) is lower bound when \( y \) is sufficiently large. This is implicitly defined by

\[
(18) \quad p^\infty(x) = \min \left\{ p \in \mathbb{R}_+ \left| \pi m \int_{P(y)} p D(p) \left( 1 - \frac{x}{D(z)} \right) d\phi^*_a(z) \right. \right\},
\]

which is defined for all \( x \geq 0 \), and is twice continuously differentiable for all \( x \in (0, X) \). Let \( \gamma(x) = D(p^\infty(x)) \), and note that \( \gamma(x) \geq \gamma(0) \) for all \( x \geq 0 \).

Let \( p(x, y) \) denote the lowest price in the support when \( y < \gamma(x) \). Formally, \( \rho(x, y) \) is defined by

\[
(19) \quad p(x, y) = \min \left\{ p \in \mathbb{R}_+ \left| \pi m \int_{P(y)} p y \left( 1 - \frac{x}{D(z)} \right) d\phi^*_a(z) \right. \right\}.
\]

The lower bound of the support is defined by

\[
(20) \quad \rho(x, y) = \begin{cases} 
    p(x, y) & \text{if } y \in (0, \gamma(x)] \\
    p^\infty(x) & \text{if } y \geq \gamma(x).
\end{cases}
\]

The key difference between the efficient and proportional rationing rules is that, under the latter, the larger firm receives a much better residual demand; hence it earns a higher expected revenue.
Under proportional rationing, the sequential pricing game does not yield the same expected revenue as the simultaneous game.\textsuperscript{9} Regardless, using the sequential pricing game is still fundamental to the proofs of the primary results of this section. The equilibrium sequential pricing game is described in section 6.2 of the Appendix.

4.2 The Capacity Choice Game

The capacity choice profit function for firm \( a \) is given by

\[ \Pi^p_a(x, y) = R^p(x, y) - a(x), \]

and the corresponding capacity choice best response of firm \( a \) to any capacity \( y \) of its rival is given by

\[ r^p_a(y) = \operatorname{arg\ max}_{x \in [0, \bar{X}]} \Pi^p_a(x, y). \]

Note that the function \( \Pi^p_a(x, y) \) is bounded and continuous for all \((x, y) \in \mathbb{R}^2_+\). Hence, for all \( y \in [0, \bar{X}] \), the set \( r^p_a(y) \) is non-empty and closed.

4.3 More Judo Economics in Action

We analyze the capacity choice game with proportional rationing using the same methodology we employed with efficient rationing. Recall that, under efficient rationing, the Judo Principles 1–3 always hold; and they yield Theorem 1. By contrast, under proportional rationing, these principles do not always hold nor does the K&S property always hold. This is no coincidence. We will show that when the appropriate analogues of the Judo Principles do hold, the K&S result also holds — even under proportional rationing.

Before concerning ourselves with the Judo Principles, we first establish when the Cournot outcome is an equilibrium. Clearly, the Cournot outcome can be an equilibrium of the capacity choice game with proportional rationing only if \((x^*, y^*) \in C\), that is, only if\textsuperscript{10}

\[ x^* + y^* \leq x^m. \]

\textsuperscript{9}Deneckere and Kovenock (1992) discuss this point in their Section 6. The difference in expected revenue is based on the fact that a firm’s residual demand is dependent on the other firm’s pricing strategy.

\textsuperscript{10}By “Cournot outcome,” we mean that both capacities and prices are the same as in the Cournot equilibrium.
By contrast, under efficient rationing the Cournot equilibrium always lies within the Cournot region $C$ — without the need for any additional condition like (1). Interestingly, the above necessary condition is also sufficient for existence.

**Theorem 2 (Existence)** $(x^*, y^*)$ is a Nash equilibrium of the capacity choice game with proportional rationing if (2) holds.

The proof of Theorem 2 is fairly extensive and we devote a subsection of the appendix to its exposition. There is no obvious direct method to prove this result. Instead, we utilize a method based on the judo game with sequential pricing. Our approach is to first prove the statement of Theorem 2 for the sequential pricing game, which is considerably more tractable. Then we use an insight from Deneckere and Kovenock (1992) to relate the judo game to the simultaneous game: at fixed capacities, the equilibrium expected revenue of each firm is weakly less in the simultaneous pricing game than in the sequential pricing game with the smaller firm pricing first. It follows that if there is no profitable deviations from $(x^*, y^*)$ with sequential pricing, then this also must be true with simultaneous pricing.

We now turn to the uniqueness question. Once again, analysis of the game with sequential pricing greatly advances our understanding of the game with simultaneous pricing. Consequently, the intractability of the expected revenues can be somewhat skirted and we are able to prove the key properties of each firm’s capacity choice best response.

The following condition is useful for extending our dominance-solvable results to proportional rationing

$$(\ddagger) \quad \min\{a'(x^*), b'(y^*)\} \geq p^m.$$  

Condition (2) is more restrictive than conditions (1) and it should be obvious that (2) implies (1).

We expand that each firm’s best response has the Judo properties for the special case that (2) holds. Most importantly, (2) allows us to establish that the best response correspondence is monotone nonincreasing.

**Lemma 2** Suppose conditions (2), the best response correspondence of an arbitrary firm in the proportional capacity choice game is as follows,

$$r^p_n(y) \in \begin{cases} r_n(y) & \text{if } y \in [0, y^*], \\ \{x \in \mathbb{R}_+ | x \leq x^*\} & \text{if } y \in [y^*, \alpha]. \end{cases}$$
The character of the best response correspondence described by Lemma 2 is shown graphically in Figure 5.

![Figure 5: The best response correspondence $r_p(y)$](image)

The primary content of the proof of Lemma 2 is broken into two parts: part 1 shows that the best response is equivalent to the Cournot best response for all $y \in [0, y^*]$ and, part 2 establishes that there is no best response greater than $x^*$ for all $y \in [y^*, X]$.

The following theorem shows that when judo is effective (when $(\dagger)$ holds), the Cournot equilibrium is the dominance-solvable solution of the capacity choice game. That is, $(\dagger)$ is sufficient for extending the K&S result to proportional rationing.

**Theorem 3 (dominance-solvability)** Under condition $(\dagger)$, $(x^*, y^*)$ is the dominance-solvable solution for the capacity choice game.

The proof follows the same lines as the proof of Theorem 1. Our argument for dominance-solvability is based the monotonic decreasing properties of the best responses, which allows us to again appeal to the results of Milgrom and Shannon (1994).

**Proof of Theorem 3.** Theorem 2 implies $(x^*, y^*)$ is a Nash equilibrium of the capacity choice game. We need only verify that $(x^*, y^*)$ is the unique pure strategy equilibrium. The proof will closely resemble the proof of Part 1 in
Theorem 1. There cannot be an equilibrium with mixed pricing because one firm would have to be in $M_a^L$. Also, any capacities $(x, y) \in B$ cannot be an equilibrium because $x^o = 0$ would be a profitable deviation. Therefore, there can only be an equilibrium $(x, y) \in C$. Since $(x^*, y^*)$ is the unique Cournot equilibrium, any other $(x, y) \in C$ must have a profitable deviation in the Cournot game. But any deviation from $(x, y)$ yields at least as much profit in the capacity choice game as in the Cournot game. Therefore, $(x^*, y^*)$ also must be the unique equilibrium of the former game.

Finally, since Lemma 2 shows that the best response correspondences are monotonic nonincreasing, the proof that the capacity choice game is dominance-solvable for $(x^*, y^*)$ is identical to the argument in Part 3 of the proof of Theorem 1.

The fact that the capacity choice game has $(x^*, y^*)$ as its unique equilibrium (even including the possibility of mixed-strategy equilibria), follows immediately from Theorem 3.

**Corollary 2** Suppose condition (†), $(x^*, y^*)$ is the unique Nash equilibrium of the capacity choice game.

So far we have focused on the proportional rationing rule. The following remark significantly widens our scope.

**Remark 2** Theorems 2 and 3 can be extended to any rationing rule with equilibrium expected revenue between the efficient and proportional rules. For any such intermediate rule, the expected revenue will always be weakly less than the expected revenue with proportional rationing. And the Cournot pricing region of capacities, which we label $C'$, will contain the proportional Cournot region $C \subset C'$. Thus, (†) immediately implies that Theorem 2 continues to hold.

Similarly under (‡), the proof of Lemma 2 can be easily extended to any such rationing rule. Since Theorem 3 follows almost immediately from Lemma 2, the theorems must still hold.

### 4.4 Examples

The following examples assume proportional rationing. They illustrate parameter specifications such that the K&S property holds — even in the absence of efficient rationing. The examples also illustrate how variations in cost functions affect the range of parameter specifications under which the K&S prop-
property holds. We begin with the simplest parameter family, linear demand and costs.

**Example 1** (linear demand and linear costs): \( P(q) = M - q, a(x) = ax, b(y) = by. \)

For this family, condition (†) simplifies to
\[
a + b \geq M/2.
\]

For a fixed value of \( M > 0 \), Figure 6 graphs this region in cost space. The large square delimits the admissible cost parameters:
\[
(a, b) \in (0, M) \times (0, M).
\]

Condition (†) holds for all admissible cost pairs in the shaded area of the figure. It is easy to see that (†) — and hence a weak version of the K&S property — will hold for most admissible parameter pairs. More precisely, within the admissible region, the parameter pairs satisfying (†), account for 7/8 of the total area. Further, among the admissible symmetric pairs (along the 45° line), the pairs satisfying (†), account for 3/4 of the length of the line segment, a surprisingly large number of parameter specifications.

The sufficient condition (‡) for the two Judo Properties, and hence for uniqueness, reduces to
\[
\min\{a, b\} \geq M/2.
\]

Graphically, this is the white diamond patterned square in the upper right corner of the figure. To be precise, the unique equilibrium of the capacity choice game will coincide with the Cournot outcome in 1/4 of the admissible region. Further, focusing on firms with symmetric costs, (‡) will be satisfied for 1/2 of all admissible symmetric parameter choices.

**Example 2** (linear demand and strictly convex costs): \( P(q) = M - q, a(x) = ax^2, b(y) = by^2. \)

The condition (‡) now simplifies to
\[
ab \geq 1/4.
\]

This region is plotted in Figure 7, consisting of the entire shaded region on or above the curved line within the positive orthant. In this example, the admissible cost parameter space is \((0, \infty) \times (0, \infty)\), a space with infinite area. Hence, except for a negligible subset of the admissible parameter pairs, the
Cournot outcome will be an equilibrium of the capacity choice game.\footnote{It is important that we clarify what is meant by “negligible.” If we fix the marginal cost parameter of one firm at any $b$, then the length (Lesbegue Measure on $\mathbb{R}$) of admissible parameters $a$ that do not lead to the K&S result is finite while the length of admissible parameters that lead to the K&S result is infinite. Thus, for a fixed $b$, the ratio of parameters which do not give the K&S result over those that do is arbitrarily small. This is what is meant by negligible.}

The sufficient condition (‡) for the capacity choice game to have the Cournot outcome as its unique equilibrium reduces to

$$\min\{ab - a, ab - b\} \geq \frac{3}{4}.$$ 

Graphically, this consists of all pairs $(a, b)$ on or above the second curve within the positive orthant of Figure 7.

When compared to the previous example, this example illustrates that increasing the convexity of costs is likely to increase the percentage of cost parameterizations such that the K&S property holds.

Figure 6: Linear demand and linear costs
5 Conclusion

We have deepened our understanding of the K&S result by generalizing to asymmetric costs and across rationing rules, using an analytic methodology based on intuitions from judo economics. By focusing on subgame perfect equilibria and hence on the rolled-back capacity choice game, we have seen that the Cournot and capacity choice games are even more closely related than one might have guessed, since both games are dominance solvable. Further, we have shown that even under proportional rationing, the K&S result holds if costs of capacity are high enough. In our baseline example (linear demand and linear costs), the K&S result holds for 1/2 of all admissible symmetric costs and over 1/4 of all admissible costs.

This judo methodology can be extended and used to analyze other perturbations of the Cournot and two-stage games. Notably:

Demand and/or cost uncertainty Under what conditions will the K&S property hold when firms face uncertainty about market demand or their rival’s costs? This question can be answered using extended versions of the three judo principles.\footnote{Reynolds and Wilson (2000) characterize when the weak K&S property will hold under}
Oligopoly rather than duopoly. Under what conditions will the K&S property continue to hold when there are more than 2 rivals? We make the educated conjecture that generalizations of the judo principles also can be used to answer this question.\footnote{There is an existing literature on oligopoly two-stage games. Boccard and Wauthy (2000, 2004) have characterized the equilibria of an oligopoly two-stage game where capacity is an imperfect commitment device. We make an educated conjecture that the methodology used in this paper could help to further expand this literature.}

Such extensions would further add to our understanding of the robustness and non-robustness of the Cournot model as a reduced form representation of the two-stage game.

6 Appendix

6.1 Efficient Rationing

Throughout appendix 6.1 we denote $p_x^\infty = p_x^\infty(x)$.

Proof of Proposition 1. We begin by showing that their is a unique maximizer of $\Pi_a(x, y)$ on $[0, X]$ (without the constraint $x \geq y$). Let us start by calculating $\partial_x \Pi_a(x, y)$:\footnote{For any function $f$ on the real line, we use $f'(x)$ to denote its derivative at the point $x$. For any function $f$ of two variables, $\partial_x f(x, y)$ will denote its partial derivative with respect to the first argument evaluated at the point $(x, y)$.}

\begin{align}
\partial_x \Pi_a(x, y) &= \frac{r'(x)x}{y} \cdot \left\{ \frac{[P'(x + r(x))r(x) + P(x + r(x))]}{y} \right\} \\
&\quad + \frac{r(x)}{y} \cdot [P'(x + r(x))x + P(x + r(x))] - a'(x) \\
&= \frac{r(x)}{y} \cdot [P'(x + r(x))x + P(x + r(x))] - a'(x).
\end{align}

The term in the expression on the first line is equal to zero based on the definition of $r(x)$ as the Cournot best response function of a firm with zero cost.

\cite{Lepore2009} uses the insights developed here to expand the analysis. Lepore (2008) has applied some of the results to analyze a model with incomplete cost information.
Denote by $\tilde{x}$, a maximizer of $\Pi_a(x, y)$. Based on the differentiability of $\Pi_a$, at a maximum it is necessary that $\partial_x \Pi_a(\tilde{x}, y) = 0$, hence,

\begin{equation}
\frac{r(\tilde{x})}{y} \cdot [P'(\tilde{x} + r(\tilde{x}))\tilde{x} + P(\tilde{x} + r(\tilde{x})) - a'(\tilde{x})] = 0.
\end{equation}

Take any $\tilde{x} \in [0, \bar{X}]$ that satisfies the equality. For any $x^o \in [0, \tilde{x})$ and any $x^1 \in (\tilde{x}, \bar{X}]$. We argue in what follows that the following set of inequalities must hold.

\begin{align}
\text{(27)} 
& P'(x^o + r(x^o))x^o > P'(\tilde{x} + r(\tilde{x}))\tilde{x} > P'(x^1 + r(x^1))x^1 \\
\text{(28)} 
& P(x^o + r(x^o)) > P(\tilde{x} + r(\tilde{x})) > P(x^1 + r(x^1)) \\
\text{(29)} 
& r(x^o) > r(\tilde{x}) > r(x^1), \\
\text{(30)} 
& -a'(x^o) > -a'(\tilde{x}) > -a'(x^1).
\end{align}

In order to verify the inequalities in (27), we show that $P'(r(x) + x)x$ is strictly decreasing in $x$ for $x$ such that $r(x) > 0$. We show the expression is strictly decreasing by differentiating in terms of $x$:

\begin{equation}
\text{(31)} 
P''(r(x) + x)[r'(x) + 1]x + P'(r(x) + x)x.
\end{equation}

Based on Assumption 1, $P'(r(x) + x) < 0$ and $P''(r(x) + x) < 0$. Adding the insight from Fact 1 that under these conditions $r'(x) < -1$, expression (31) must be negative for all $x$ such that $r(x) > 0$.

The inequalities in (28) are based on the fact $P(x)$ is strictly decreasing on $x \in [0, \bar{X}]$ and that from Fact 1(b) that $x^o + r(x^o) < \tilde{x} + r(\tilde{x}) < x^1 + r(x^1)$.

The inequalities in (29) follow directly from Fact 1(b).

The final set of inequalities (30), follow from Assumption 2; that the cost functions are differentiable, increasing and convex.

Putting the above inequalities together with (26): For any $x^o \in [0, \tilde{x})$ and any $x^1 \in (\tilde{x}, \bar{X}]$,

\begin{align}
& \frac{r(x^o)}{y} \cdot [P'(x^o + r(x^o))x^o + P(x^o + r(x^o))] - a'(x^o) > 0, \\
& \frac{r(x^1)}{y} \cdot [P'(x^1 + r(x^1))x^1 + P(x^1 + r(x^1))] - a'(x^1) < 0.
\end{align}

\textsuperscript{15}Based on the continuity of the function $\Pi_a$ and the fact that the constraint set $[0, \bar{X}]$ is compact, such a maximizer must exist.
Thus, \( \tilde{x} \) is the unique maximizer of \( \Pi_a(x, y) \).

Notice that for the constrained maximization program if \( \tilde{x} \in [0, y] \), then \( x_a(y) = \tilde{x} \). But if \( \tilde{x} \in (y, \bar{X}] \), then the positive first derivative for all \( x^o \in [0, \tilde{x}] \) implies that the maximizer in the set will be the largest admissible capacity, i.e., \( x_a(y) = y \).

Next we move to showing that \( x_a \) is the unique maximizer of \( \Pi_a^\infty(x) \). The first derivative of \( \Pi_a^\infty(x) \) is found using implicit differentiation. Define the expression \( V(p, x) = pD(p) - P(r(x) + x)r(x) \). Then the first derivative of \( \Pi_a^\infty(x) \) is \( \partial_x \Pi_a^\infty(x) = \partial_x p_x^\infty \cdot x + p_x^\infty - a'(x) \), where, \( \partial_x p_x^\infty = -\partial_x V(p, x)/\partial_p V(p, x) \).

\[
\partial_x \Pi_a^\infty(x) = r(x) \cdot \left[ \frac{P'(x + r(x))}{D'(p_x^\infty)p_x^\infty + D(p_x^\infty)} + \frac{P(x + r(x))}{D(p_x^\infty)} \right] - a'(x).
\]

Based on the differentiability of \( \Pi_a^\infty \), at a maximum it is necessary that \( \partial_x \Pi_a^\infty(x_a) = 0 \). Let us take an \( x_a \in [0, \bar{X}] \) that satisfies the equality \( \partial_x \Pi_a^\infty(x_a) = 0 \). For any \( x^o \in [0, x_a] \) and any \( x^1 \in (x_a, \bar{X}] \), the set of inequalities above (27 – 30) are still evident. In what follows, we argue the two sets of inequalities below must hold.

\[
\begin{align*}
D'(p_x^\infty)p_x^\infty &< D'(p_x^\infty)p_x^\infty < D'(p_x^\infty)p_x^\infty, \\
D(p_x^\infty) &< D(p_x^\infty) < D(p_x^\infty).
\end{align*}
\]

Now verify the inequalities in (33). First, notice that based on Assumption 1, \( D'(p)p < 0 \) and is strictly decreasing for all \( p \in [0, p^m] \) and the fact that \( p_x^\infty \) must be in \( [0, p^m] \). Next we argue that \( p_x^\infty \) is weakly less than \( p^m \) strictly decreasing in \( x \) on \( [0, \bar{X}] \) based on the definition of \( p_x^\infty \) in equation (5): The right-hand side of equation (5) is strictly decreasing in \( x \) on \( [0, \bar{X}] \) and independent of \( p \), while the left-hand side is strictly increasing in \( p \) on \( [0, p^m] \), based on Assumption 1. Thus, as \( x \) increases the right-hand side of (5) decreases and to preserve the equality the left-hand side must decrease, which can only happen if \( p \) decreases. Also, notice that this implies the largest \( p_x^\infty \) will be when \( x = 0 \) and plugging in this to equation (5) yields \( p_0^\infty = p^m \). Thus, \( p_x^\infty \) must be in \( [0, p^m] \) for all \( x \in [0, \bar{X}] \).

The inequalities in (34) also follow from Assumption 1 and the fact that \( p_x^\infty \) must be in \( [0, p^m] \) and is strictly decreasing in \( x \) on \( [0, \bar{X}] \).

\[\text{Based on the continuity of the function } \Pi_a^\infty \text{ and the fact that the constraint set } [0, X] \text{ is compact, such a maximizer must exist.}\\
\]
Putting the list of inequalities (27 – 30) and (33 – 34) together with the fact that \( \partial_x \Pi_a^\infty(x_a) = 0 \): For any \( x^o < x_a \) and any \( x^1 > x_a \)

\[
\begin{align*}
    r(x^o) & \cdot \left[ \frac{P'(x^o + r(x^o))x^o}{D'(p^\infty_{x^o})p^\infty_{x^o} + D(p^\infty_{x^o})} + \frac{P(x^o + r(x^o))}{D(p^\infty_{x^o})} \right] - a'(x^o) > 0, \\
    r(x^1) & \cdot \left[ \frac{P'(x^1 + r(x^1))x^1}{D'(p^\infty_{x^1})p^\infty_{x^1} + D(p^\infty_{x^1})} + \frac{P(x^1 + r(x^1))}{D(p^\infty_{x^1})} \right] - a'(x^1) < 0.
\end{align*}
\]

Thus, \( x_a \) is the unique maximum of \( \Pi^\infty_a(x) \).

**Proof of Lemma 1.** The proof proceeds in three parts. Part 1 proves the 1st *Judo Principle* that it is never optimal to be the victim of judo. The second part proves the 2nd *Judo Principle*, that a firm will only use judo if the rival firm is larger than \( y \). The final part proves the 3rd *Judo Principle*, that a judo player is not too big or too small, and also provides the exact character of the best response correspondence for \( y \in [y, \bar{X}] \). Together these three parts provide the characterization of \( r^e_a(y) \) in Lemma 1 and that \( r^e_a(y) \) is monotonically nonincreasing.

Let us begin with the proof of Part 1.

**Part 1 (1st Judo Principle)** In this part we show that for all \( y \in [0, \bar{X}] \), the capacity choice best response \( r^e_a(y) \) will never be such that \( \exists x \in r^e_a(y) \) and \( (x, y) \in M^L_a \). Suppose to the contrary that for some \( y \in [0, \bar{X}] \), \( \exists x \in r^e_a(y) \) such that \( (x, y) \in M^L_a \), i.e. \( x \geq y \) and equilibrium pricing is in mixed strategies. We will show a contradiction.

Let us begin by looking at the Cournot best response function of a zero cost firm, which defines the boundaries between the pricing regions \( C \) and \( M^S_a \) or \( M^L_a \). If \( r(y) \geq y \), then \( (r(y), y) \in C \). While, if \( r(y) < y \), then \( (r(y), y) \in M^S_a \).

Let us establish that \( \forall (x, y) \in M^S_a \), \( R^e(x, y) \geq P(x + y)x \). Suppose to the contrary that \( R^e(x, y) < P(x + y)x \), which is true if and only if \( p^J(x, y) < P(x + y) \). We will show a contradiction. Let us begin with the following inequality based on the fact that \( r(x) \) is the zero cost Cournot best response.

\[
P(x + y)y \leq P(r(x) + x)r(x) = p^J(x, y) \min\{D(p^J(x, y)), y\} \leq p^J(x, y)y.
\]

The second equality is based on the definition of \( p^J(x, y) \), while the final inequality is obvious. Thus, \( p^J(x, y) < P(x + y) \) a contradiction.

Based on the preceding argument, for both cases \( (r(y), y) \in C \) and \( (r(y), y) \)
\[ \Pi_a^e(r(y), y) \geq P(r(y) + y)r(y) - a(r(y)) \]

For all \( y \in [0, \bar{X}] \) and \( x \) such that \((x, y) \in M_a^L \), firm \( a \)'s expected profit in the capacity choice game is

\[ \Pi_a^e(x, y) = P(r(y) + y)r(y) - a(x). \]

Based on the fact that \((x, y) \in M_a^L, x > r(y) \). By Assumption 2, the cost function is increasing, for all \( y \in [0, \bar{X}] \) and \( x \) such that \((x, y) \in M_a^L \). Hence, \( \Pi_a^e(x, y) < \Pi_a^e(r(y), y) \), a contradiction.

**Part 2 (2nd Judo Principle)** In this part of the proof we show that for all \( y \in [0, \bar{y}] \) the best response of the capacity choice game is equivalent to the Cournot best response function, \( r_a^*(y) = r_a(y) \). (Recall that \( y \) is defined implicitly as the unique capacity that satisfies \( y = r(r_a(y)). \)

Define \( r^* \) as the unique capacity that satisfies \( r^* = r(r(r^*)) \). If \( y \in [0, r^*] \), then equilibrium pricing can only be in region \( C \) or \( M_a^L \). For all \( y \in [0, r^*] \) and \( x \in [0, r(y)], (x, y) \in C \) (notice that \( r(y) \geq r^* \)), while for all \( y \in [0, r^*] \) and \( x \in [r(y), \bar{X}], (x, y) \in M_a^L \). Based on the proof of Part 1, for all \( y \in [0, r^*] \), \( x \in r_a^*(y) \) must be such that \((x, y) \in C \). Hence, from the proof of Part 1 we already know that \( r_a^e(y) = r_a(y) \), for all \( y \in [0, r^*] \).

To complete the proof Part 2 we are left to show that \( r_a^e(y) = r_a(y) \), for all \( y \in (r^*, \bar{y}] \). Again using the result of Part 1, showing that \( r_a^e(y) = r_a(y) \) for all \( y \in (r^*, \bar{y}] \) can be reduced to showing that \( \exists x \in r_a^e(y) \) such that \((x, y) \in M_a^S \), for all \( y \in (r^*, \bar{y}] \).

Recall that for all \((x, y) \in C \), \( \Pi_a^e(x, y) = \Pi_a^e(x, y) \). Next we show that \((x, y) \in M_a^S, \Pi_a(x, y) = \Pi_a^e(x, y) \) by establishing that for all \((x, y) \) at the boundary of \( C \) (i.e. such that \( y > x \) and \( r(x) = y \), \( y < y(x) \)). Then it follows from the definitions of \( \Pi_a^e, \Pi_a \) and \( \Pi_a \), that \( \Pi_a^e(r^{-1}(y), y) = \Pi_a(r^{-1}(y), y) = \Pi_a^e(r^{-1}(y), y) \) for all \( y \in (r^*, \bar{y}] \).

Let us now establish that for all \((x, y) \) at the boundary of \( C, y < y(x) \). By definition,

\[ p(x, y)y = P(r(x) + x)r(x) \]

Since \( r(x) = y \), (35) reduces to \( p(x, y) = P(r(x) + x) \).

Now let us suppose to the contrary that \( y \geq y(x) \). If \( y \geq y(x) \), then \( p^\infty(x) \geq p(x, y) = P(r(x) + x) \) and

\[ D(p^\infty(x)) = D(P(r(x) + x)) = r(x) + x. \]
Thus,
\[ p^\infty(x)D(p^\infty(x)) \geq P(r(x) + x)(r(x) + x) \]
\[ > P(r(x) + x)r(x), \]
which contradicts the defining equality of \( p^\infty(x) \). Therefore for all such \((x, y)\), \( y < y(x) \).

We proceed by showing that for all \( y \in (r^*, y] \) and \( x \) such that \((x, y) \in M^S_a \),
\[ \partial_x \Pi^c_a(x, y) = \partial_x \Pi_a(x, y) < 0, \]
which implies that such an \( x \) cannot be a best response to such a \( y \). This is done in a few steps.

We first show that \( \partial_x \Pi_a(r^{-1}(y), y) = \partial_x \Pi^c_a(r^{-1}(y), y) \). For an arbitrary \( x \), based on equation (25) we have:

\[ \partial_x \Pi_a(x, y) = \frac{r(x)}{y} \cdot [P'(x + r(x))x + P(x + r(x))] - a'(x). \]

Substituting \( x = r^{-1}(y) \) and using the fact that \( r(r^{-1}(y)) = y \) yields:

\[ \partial_x \Pi_a(r^{-1}(y), y) = P'(r^{-1}(y) + y)r^{-1}(y) + P(r^{-1}(y) + y) - a'(r^{-1}(y)) \]
\[ = \partial_x \Pi^c_a(r^{-1}(y), y), \]
as was to be shown.

Based on the definition of \( C \), \((r^{-1}(y), y) \in C \) is on the boundary of \( C \). Thus, \( \partial_x \Pi_a(r^{-1}(y), y) = \partial_x \Pi^c_a(r^{-1}(y), y) \) implies that \( \Pi^c_a \) is differentiable at \((r^{-1}(y), y) \) and \( \partial_x \Pi_a(r^{-1}(y), y) = \partial_x \Pi^c_a(r^{-1}(y), y) \).

Next we argue that \( r_a(y) \leq r^{-1}(y) \) for all \( y \in (r^*, y] \). Recall that \( y \) is the unique Cournot equilibrium quantity of a zero cost firm in competition against a firm with cost \( a \), i.e. \( y = r(r_a(y)) \). Notice that \( r^{-1}(y) = r_a(y) \) and \( r^{-1}(0) = \bar{X} > x^m \geq r_a(0) \). Therefore, \( r^{-1}(0) - r_a(0) > 0 \) and \( r^{-1}(y) - r_a(y) = 0 \). Let us suppose to the contrary that \( r_a(y) > r^{-1}(y) \) for some \( y \in (r^*, y] \), which implies that \( r^{-1}(y) - r_a(y) < 0 \). Since \( r^{-1}(y) - r_a(y) \) is a continuous function, the intermediate value theorem implies that \( r^{-1}(y) - r_a(y) = 0 \) for some \( y \in (0, y] \). Thus, a second Cournot equilibrium must exist in the Cournot game, a contradiction.

We now show that, for any \( y \in (r^*, y] \) and \( x \) such that \((x, y) \in M^S_a \), \( \partial_x \Pi^c_a(x, y) < 0 \).
Based on the definition of $C$, $(r^{-1}(y), y) \in C$. Notice that at $r^{-1}(y)$,

$$\partial_x \Pi_a^e(r^{-1}(y), y) = \partial_x \Pi_a^e(r^{-1}(y), y) \leq 0$$

because $r_a(y) \leq r^{-1}(y)$ and $\Pi_a^e$ is strictly concave.

For any $x > r^{-1}(y)$,

$$\partial_x \Pi_a(x, y) = \frac{r(x)}{y} \cdot [P'(x + r(x))x + P(x + r(x))] - a'(x)$$

$$< \partial_x \Pi_a^e(r^{-1}(y), y).$$

The inequality on the second line follows from $r(x)/y < 1$ and is based on Assumptions 1&2; that both $P'(x + r(x))x + P(x + r(x))$ and $-a'(x)$ are decreasing functions of $x$ for any $x > r^{-1}(y)$.

Putting the above steps together, we know that for all $y \in (r^*, \bar{y}]$ and $x \in (r^{-1}(y), y)$,

$$\partial_x \Pi_a^e(x, y) = \partial_x \Pi_a(x, y)$$

$$< \partial_x \Pi_a^e(r^{-1}(y), y)$$

$$\leq 0,$$

a contradiction to $x \in (r^{-1}(y), y)$ as a best response to any $y \in (r^*, \bar{y}]$. Therefore, $r_a^e(y) = r_a(y)$ for all $y \in [0, \bar{y}]$.

**Part 3 (3rd Judo Principle)** In Parts 1 and 2 of this proof we have established that $r_a^e(y) = r_a(y)$ for all $y \in [0, \bar{y}]$. Further, based on equation (37) and the fact that $r_a(y) = r^{-1}(y)$, $r_a^e(y) = r_a(y) = x_a(y)$. Since $\bar{y}$ is such that $r_a(y) = r^{-1}(y)$,

$$\partial_x \Pi_a(r_a(y), y) = \partial_x \Pi_a^e(r_a(y), y) = \partial_x \Pi_a^e(r_a(y), y) = 0.$$

Therefore, $r_a^e(y) = r_a(y) = x_a(y)$. Since we established that $y > y(r_a(y))$ in Part 2, $r_a^e(y)$ is continuous at $\bar{y}$ and we are only left to address $r_a^e(y)$ for all $y \in (\bar{y}, \bar{X}]$.

In this part we show that there exists $\bar{y}$ such that $r_a^e(y) = x_a(y) = x_a(y) < r_a^e(y)$ for all $y \in (\bar{y}, \bar{X}]$, $r_a^e(y) = x_a(y)$ for all $y \in (\bar{y}, \bar{X}]$ and $r_a^e(y) = \{x_a(y), x_a\}$. We also show that $x_a(y)$ is strictly decreasing in $y$, $x_a(y) < r_a^e(y)$ for all $y \in (\bar{y}, \bar{y}]$, and $x_a(y) < x_a(y)$, which together make the best response monotonic nonincreasing on $(\bar{y}, \bar{X}]$.

To begin we argue that $r_a(y) > r^{-1}(y)$ for all $y \in (\bar{y}, \bar{X}]$. Again recall that $\bar{y}$ is the unique Cournot equilibrium quantity of a zero cost firm in competition.
against a firm with cost \( a \), i.e. \( y = r(r_a(y)) \). Notice that \( r^{-1}(y) = r_a(y) \) and \( r_a(x^m) > r^{-1}(x^m) = 0 \). Therefore, \( r^{-1}(0) = r_a(0) > 0 \) and \( r^{-1}(y) - r_a(y) = 0 \).

Let us suppose to the contrary that \( r_a(y) \leq r^{-1}(y) \) for some \( y \in (y, \bar{X}] \), which implies that \( r^{-1}(y) - r_a(y) \leq 0 \). Since \( r^{-1}(y) - r_a(y) \) is a continuous function, the intermediate value theorem implies that \( r^{-1}(y) - r_a(y) = 0 \) for some \( y \in (y, \bar{X}] \). Thus, a second Cournot equilibrium must exist in the Cournot game, a contradiction.

Observe that for all \( y \in (y, \bar{y}] \), \( r^e_a(y) \geq r^{-1}(y) \) based on \( r_a(y) \geq r^{-1}(y) \).

The strict concavity of \( \Pi^e_a \) implies that if firm \( a \) is to choose \( x \) in the Cournot pricing region, then its highest profit will be at \( r^{-1}(y) \). Based on Part 2, we know that \( \partial_x \Pi_a(r^{-1}(y), y) = \partial_x \Pi_a(r^{-1}(y), y) \) for all \( y \in (y, \bar{X}] \). Hence, \( \partial_x \Pi_a(r^{-1}(y), y) < 0 \), for all \( y \in (y, \bar{X}] \). Thus, there exists an \( x_a(y) \) that leads to greater profit than \( r^{-1}(y) \). Hence, for all \( y \in (y, \bar{X}] \) and \( x \in r^e_a(y) \), \( (x, y) \in M^S_a \), which implies that \( r^e_a(y) \subset \{ x_a(y), x_a \} \) for all \( y \in (y, \bar{X}] \).

Next we show that for all \( y \in (y, \bar{X}] \) and \( x \in r^e_a(y), x \leq r_a(y) \). This is done in two steps. Step (i) shows that for all \( y \geq y \), the capacity \( x_a(y) \) cannot exceed \( r_a(y) \). Step (ii) shows that there exists a unique \( \bar{y} \in (y, \bar{X}] \) such that \( \Pi_a(x_a(\bar{y}), \bar{y}) = \Pi^\infty_a(x_a), \Pi_a(x_a(y), y) > \Pi^\infty_a(x_a) \) for all \( y \in [y, \bar{y}] \) and \( \Pi_a(x_a(y), y) < \Pi^\infty_a(x_a) \) for all \( y \in (\bar{y}, \bar{X}] \) and \( x_a < x_a(y) \).

Step (i): We establish the upper bound of the best response \( \forall y \geq y, x_a(y) \leq r_a(y) \). Based on the argument for single peakedness of \( \Pi_a \) in Proposition 1, \( \partial_x \Pi_a(x, y) < 0 \), for all \( x > r_a(y) \) such that \( (x, y) \in M^S_a \). We are left to show that \( \partial_y \Pi_a(x, y) \) is decreasing in \( y \), for all \( y \geq y \) and \( x > r_a(y) \) such that \( (x, y) \in M^S_a \), or the marginal expected revenue of firm \( a \) is negative.

If the marginal expected revenue is negative, then the marginal expected profit is as well, and \( x \) cannot be a best response for any \( y \). If the marginal expected revenue of firm \( a \) is positive, then \( y \) is the divisor of the only positive term in the marginal expected profit expression (26). Hence, \( \partial_y \Pi_a(x, y) < 0 \) for all \( x > r_a(y) \) and \( y \geq y \) such that \( (x, y) \in M^S_a \).

Step (ii): We complete the proof of Part 3 by showing that there exists a unique \( \bar{y} \in (y, \bar{X}] \) such that \( \Pi_a(x_a, \bar{y}) = \Pi^\infty_a(x_a), \Pi_a(x_a(y), y) > \Pi^\infty_a(x_a) \) for all \( y \in [y, \bar{y}] \), and \( \Pi_a(x_a(y), y) < \Pi^\infty_a(x_a) \) for all \( y \in (\bar{y}, \bar{X}] \) and \( x_a < x_a(y) \).

Recall \( x_a \) is the optimizer of \( \Pi^\infty_a(x) \). Suppose \( y = D(p^\infty_{x_a}) \). Based on equations (25) and (32), the first order necessary conditions for a maximum
of \( \Pi_a(x, y) \) and \( \Pi_a^\infty(x) \) in terms of \( x \) are:

\[
(38) \quad r(x) \cdot \left[ \frac{P'(r(x) + x)}{y} + \frac{P(r(x) + x)}{y} \right] - a'(x) = 0,
\]

\[
(39) \quad r(x) \cdot \left[ \frac{P'(x + r(x))x}{D'(p_x^\infty)p_x^\infty + D(p_x^\infty)} + \frac{P(x + r(x))}{y'} \right] - a'(x) = 0.
\]

Taking \( y' = D(p_x^\infty) \), equation (39) can be re-written as

\[
(40) \quad r(x) \cdot \left[ \frac{P'(x + r(x))x}{D'(p_x^\infty)p_x^\infty + y'} + \frac{P(x + r(x))}{y'} \right] - a'(x) = 0.
\]

Any capacity \( x_a \) that satisfies (40) must be less than a capacity \( x_a(y) \) that satisfies (38) because

\[
\frac{P'(x + r(x))x}{D'(p_x^\infty)p_x^\infty + y'} < \frac{P'(x + r(x))x}{y'} < 0
\]

and both terms are decreasing in \( x \). Hence \( x_a < x_a(y') \) at \( y' = D(p_x^\infty) \).

At \( y' = D(p_x^\infty) \), \( x_a(y') \) is the unique optimizer of \( \Pi_a(x_a(y'), y') \), which implies that profits are ordered so that

\[
\Pi_a(x_a(y'), y') > \Pi_a(x_a, y') = \Pi_a^\infty(x_a),
\]

and thus the judo safe price is \( p(x_a(y'), y') \).

Suppose \( y^o \) is such that \( x_a = x_a(y^o) \). Then, from the above necessary conditions (38) and (39), \( y^o > D(p^\infty(x_a)) \). Thus profits are ordered so that

\[
\Pi_a(x_a(y^o), y^o) = \Pi_a(x_a, y^o) < \Pi_a^\infty(x_a),
\]

and the judo safe price is \( p_{x_a}^\infty \). At some capacity between \( D(p_{x_a}^\infty) \) and \( y^o \), firm \( a \) switches from \( p(x_a(y), y) \) to \( p_{x_a}^\infty \).

Based on the continuity of \( \Pi_a(x_a(y), y) - \Pi_a^\infty(x_a) \) in \( y \) and that there exists \( y^o, y' \in (y, \bar{X}] \) such that \( \Pi_a(x_a(y^o), y^o) - \Pi_a^\infty(x_a) < 0 \) and \( \Pi_a(x_a(y'), y') - \Pi_a^\infty(x_a) > 0 \), the intermediate value theorem guarantees the existence of an \( \bar{y} \in (y', y^o) \) such that \( \Pi_a(x_a(\bar{y}), \bar{y}) - \Pi_a^\infty(x_a) = 0 \).

Next we show that \( \Pi_a(x_a(y), y) - \Pi_a^\infty(x_a) < 0 \) for all \( y \in (y, \bar{y}] \) and \( \Pi_a(x_a(y), y') - \Pi_a^\infty(x_a) < 0 \) for all \( y' \in (\bar{y}, \bar{X}] \). To do this, we show that \( \Pi_a(x_a(y), y) \) is strictly decreasing in \( y \). As a preliminary step, we show that \( x_a(y) \) strictly decreasing in \( y \) on \([0, \bar{X}] \). We can rewrite the equality in expression (25) as

\[
(41) \quad r(x)[P'(x + r(x))x + P(x + r(x))] - ya'(x) = 0.
\]
If \( y \) increases, then the left-hand side of expression (41) becomes negative. The first term is positive and based on Fact 1(b) a decrease in \( x \) increases \( r(x) \) and decreases \( x + r(x) \). Since \( P \) is concave and downward sloping (Assumption 1), \( P'(x + r(x))x \) becomes less negative and \( P(x + r(x)) \) increases based as \( x \) decreases. Thus, the first term (41) increases when \( x \) decreases. Based on Assumption 2, the convexity of \( a, ya' \) decreases as \( x \) decreases. Putting this all together, if \( y \) is increased, then the \( x \) that satisfies (41) must decrease, i.e. \( x_a(y) \) strictly decreasing in \( y \) on \([0, \bar{X}]\).

Now we show that \( \Pi_a(x_a(y), y) \) is strictly decreasing in \( y \) on \([0, \bar{X}]\). We take the derivative of \( \Pi_a(x_a(y), y) \) in terms of \( y \)

\[
\partial_y \Pi_a(x_a(y), y) = \partial_x \Pi_a(x_a(y), y)x'_a(y) + \partial_y \Pi_a(x_a(y), y)\big|_{x=x_a(y)}.
\]

The first term on the left-hand side is zero based on the definition of \( x_a(y) \) as the maximizer of \( \Pi_a(x, y) \). Calculating the remaining term based on (25), the expression can be simplified to

\[
\partial_y \Pi_a(x_a(y), y) = -\frac{1}{y}p(x_a(y), y)x_a(y) < 0.
\]

Since \( \Pi_a(x_a(y), y) - \Pi_a(x_a(y)) \) is continuous and strictly decreasing in \( y \), there must exist a unique \( \bar{y} \in (y', y^o) \) such that \( \Pi_a(x_a(\bar{y}), \bar{y}) = \Pi_a^\infty(x_a) \) at \( \bar{y} \), \( \Pi_a(x_a(y), y) > \Pi_a^\infty(x_a) \) for all \( y \in [\bar{y}, \bar{y}] \), and \( \Pi_a(x_a(y), y) < \Pi_a^\infty(x_a) \) for all \( y \in [\bar{y}, \bar{X}] \).

We established in Part 2 of the proof of this lemma that \( y(r_a(y)) > \bar{y} \). Since \( y^o > \bar{y}, x_a(y^o) = x_a \) and \( x_a(y) \) is strictly decreasing in \( y \), \( x_a(y) = x_a(y^o) = x_a \). Together with Step (i), we have shown that \( r_a(y) \) is monotonic nonincreasing on \([0, \bar{X}]\), as well as the specific characterization in the statement of the lemma.

6.2 Proportional Rationing

The structure of this section goes as follows. First, we state some key facts about the Cournot game. Second, we outline the properties of the judo sequential pricing game with proportional rationing. Third, we state and prove some key properties of the capacity choice game with sequential judo pricing. Finally, we use the tools developed in this section to prove our main results about proportional rationing.

**Key facts about the Cournot game** (proofs standard):
\((E)\) \(\partial_x \Pi_a^c(x^*, y^*) = 0.\)

\((C1)\) \(\partial_x \Pi_a^c(x, y)\) is decreasing in \(y.\)

\((C2)\) \(\Pi_a^c(x, y)\) is strictly concave in \(x.\)

\((C3)\) If \(\min\{x^*, y^*\} \leq x_m/2\), then \(\partial_x \Pi_a^c(x_m - y, y) < 0\) for all \(y < y^*.\)

We will use properties about the sequential pricing game to prove the main results regarding the simultaneous choice game. First let us define the expected revenue of the sequential pricing game where the smaller firm prices first; this follows the same lines as with efficient rationing (Section 3.1). The Nash equilibrium expected revenue for firm \(a\) in each region is given by the function:

\[
\tilde{R}(x, y) = \begin{cases} 
P(x + y)x & \forall (x, y) \in \mathbb{C} \\
\pi^m \left(1 - \frac{y}{D(p_a)}\right) & \forall (x, y) \in \mathbb{M}_a^L \\
\rho^l(x, y)x & \forall (x, y) \in \mathbb{M}_a^S \\
0 & \forall (x, y) \in \mathbb{B},
\end{cases}
\]

The correspondence \(\rho^l(x, y)\) is the judo safe price under proportional rationing, it will be defined shortly.

The larger firm will maximize its revenue taking as given the residual demand remaining after the smaller firm has sold its entire capacity. The residual revenue is always maximized by monopoly pricing \(p_m^a.\)

\[
\pi^m \left(1 - \frac{x}{D(p_a)}\right), \quad \text{for all } (x, y) \in \mathbb{M}_b^L.
\]

Let \(\rho^\infty(x)\) denote the profit maximizing price such that firm \(a\) is safe from being undercut when its rival has essentially unlimited capacity. Mathematically, this is

\[
\rho^\infty(x) = \min \left\{ p \in \mathbb{R}_+ \bigg| pD(p) = \pi^m \left(1 - \frac{x}{D(p)}\right) \right\},
\]

which is defined for all \(x \geq 0\), and is twice continuously differentiable for all \(x \in (0, X)\).

Let \(\gamma(x) = D(\rho^\infty(x))\), and note that \(\gamma(x) \geq \gamma(0)\) for all \(x \geq 0\). Firm \(b\) has “essentially unlimited capacity” means \(y > \gamma(x)\).
Let $\rho(x, y)$ denote the profit maximizing price such that firm $a$ is safe from being undercut when $y < \gamma(x)$. Formally, the $\rho(x, y)$ is defined by

\begin{equation}
\rho(x, y) = \min \left\{ p \in \mathbb{R}_+ \mid py = \pi^m \left(1 - \frac{x}{D(p)} \right) \right\}.
\end{equation}

As a function, $\rho(x, y)$ is defined for all $y > 0$ and $x \geq 0$, and is twice continuously differentiable for all $x \in (0, \bar{X})$.

The \textit{judo safe price} under proportional rationing is the maximum of the above two safe prices:

\begin{equation}
\rho^j(x, y) = \begin{cases} 
\rho(x, y) & \text{if } y \in (0, \gamma(x)] \\
\rho^\infty(x) & \text{if } y \geq \gamma(x).
\end{cases}
\end{equation}

**Facts about the sequential proportional game** (proofs provided below):

**P1** For all $y \in (0, x^m)$, $\partial_x \Pi^c_a(x^m - y, y) = \partial_x \widetilde{\Pi}^p_a(x^m - y, y)$.

**P2** for all $x^o \in (x^m - y, \bar{X}]$ and $y \in [0, x^m]$, $\partial_x \Pi^p_a(x^m - y, y) > \partial_x \widetilde{\Pi}^p_a(x^o, y)$.

To simplify notation we denote the judo prices $\rho_y^j = \rho^j(y, x)$, $\rho_y = \rho(y, x)$, and $\rho_y^\infty = \rho^\infty(y)$ throughout the rest of the appendix.

**Proof of P1.** For all $y \in (0, x^m)$, $\partial_x \Pi^c_a(x^m - y, y) = \partial_x \widetilde{\Pi}^p_a(x^m - y, y)$. We derive a simplified form of $\partial_x \Pi^c_a(x^m - y, y)$ below:

\begin{align}
\partial_x \Pi^c_a(x^m - y, y) &= P'(x^m)(x^m - y) + p^m - a'(x) \\
&= P'(x^m)x^m + p^m - P'(x^m)y - a'(x) \\
&= -P'(x^m)y - a'(x).
\end{align}

The first term in the second line equals zero based on the definition of $x^m$ and $p^m$, as the zero cost monopoly quantity and price, respectively.

For all $y \in (0, x^m/2]$, we derive a simplified form of $\partial_x \widetilde{\Pi}^p_a(x^m - y, y)$ below:

\begin{equation}
\partial_x \widetilde{\Pi}^p_a(x^m - y, y) = \frac{\pi^m y D'(\rho_y^j)}{D(\rho_y^j)^2} \cdot \partial_x \rho_y^j - a'(x)
\end{equation}

We can restrict attention to $\rho_y$ because it is straightforward to verify that $(x^m - y, y)$ cannot lead to judo pricing $\rho_y^\infty$. We find $\partial_x \rho_y$ by implicit differentiation. Recall from equation (45) that

\begin{align}
\rho_y x &= \pi^m \left(1 - \frac{y}{D(\rho_y)} \right) \\
V^p(\rho_y, x) &= D(\rho_y) \rho_y x - \pi^m D(\rho_y) + \pi^m y = 0.
\end{align}
We utilize implicit differentiation to find an expression for $\partial_x \rho_y$,

$$
\partial_x \rho_y = -\frac{\partial_x V^p(p_y, x)}{\partial_y V^p(p_y, x)} = \frac{D(p^m y)}{D(p^m) \pi^m - (D'(p^m) p^m + D(p^m)) x < 0}.
$$

Back to our derivation, it is also straightforward that $\rho_x = \rho_y = p^m$, for all $(x, y)$ such that $x + y = x^m$. Hence, we can simplify (48) by substituting in (50) and using the fact that $\rho_y = p^m$. Thus at $x + y = x^m$, $\partial_x \rho_y$ simplifies to

$$
\partial_x \rho_y = \frac{D(p^m) p^m}{D(p^m) \pi^m - (D'(p^m) p^m + D(p^m)) x = -p^m}.
$$

The entire expected revenue expression can be simplified to:

$$
\partial_x \tilde{R}^p(x^m - y, y) = \frac{\pi^m y D'(p^m)}{D(p^m)^2} \cdot \left( \frac{-p^m}{x^m} \right) = \frac{p^m y}{x^m}.
$$

For all $y \in (x^m/2, x^m)$, we show that $\partial_x \tilde{\Pi}^p(x^m - y, y) = p^m y / x^m - a'(x)$ as well. The revenue function can be expressed as,

$$
\tilde{R}^p(x^m - y, y) = \rho_x x.
$$

Once again we utilize implicit differentiation, this time to find an expression for $\partial_x \rho_y$:

$$
\partial_x \rho_y = \frac{\pi^m}{D'(p^m) \pi^m - (D'(p^m) p^m + D(p^m)) y} = -\frac{p^m}{x^m}.
$$

Now we substitute (52) into the $\partial_x \tilde{R}^p(x^m - y, y)$:

$$
\partial_x \tilde{R}^p(x^m - y, y) = \rho_x + x \cdot \partial_x \rho_x = \frac{p^m y}{x^m}.
$$

Finally we show the equality $\partial_x \Pi^p(x^m - y, y) = \partial_x \tilde{\Pi}^p(x^m - y, y)$. We start by substituting in using (47) and (51).

$$
-P'(x^m)y = \frac{p^m y}{x^m}
$$

$$
P'(x^m)x^m + p^m = 0.
$$

The last line must be true since this condition defines the monopoly solution.
Proof of P2.  \( \partial_x \tilde{\Pi}_a^p(x^m - y, y) > \partial_x \tilde{\Pi}_a^p(x^o, y) \) for all \( y \in [0, x^m] \) and \( x^o > x^m - y \).

First we show that for all \((x^o, y)\) such that the judo price is \( \rho^\infty(y), \partial_x \tilde{\Pi}_a^p(x^m - y, y) > \partial_x \tilde{\Pi}_a^p(x^o, y) \) by showing that \( \partial_x \tilde{p}(x^o, y) = 0 \).

Based on the expression found in (47), \( \partial_x \tilde{p}(x^m - y, y) = -y p'(x^m) y > 0 \) and based on the convexity of the cost function (Assumption 2), \(-a'(x^o) < -a'(x^m - y)\). Together with the two inequalities in the previous sentence, \( \partial_x \tilde{p}(x^o, y) = 0 \) implies that \( \partial_x \tilde{\Pi}_a^p(x^m - y, y) > \partial_x \tilde{\Pi}_a^p(x^o, y) \). From expression (48) we know that

\[
\partial_x \tilde{p}(x^o, y) = \frac{\pi^m y D'(\rho^\infty)}{D(\rho^\infty)^2} \cdot \partial_x \rho^\infty.
\]

Since \( x \) is not an argument of \( \rho^\infty, \partial_x \rho^\infty = 0 \), and thus \( \partial_x \tilde{p}(x^o, y) = 0 \).

Now consider \((x^o, y)\) such that the judo price is \( \rho(y, x^o) \), which we denote by \( \rho_y^o \). Using (50) we rewrite \( \partial_x \tilde{p}(x^o, y) \),

\[
(53) \quad \partial_x \tilde{p}(x^o, y) = \frac{\pi^m y D'(\rho^o_y) \rho_y^o}{D(\rho^o_y) \cdot [ D'(\rho^o_y) \pi^m - (D'(\rho^o_y) \rho_y^o + D(\rho^o_y)) x] }.
\]

We use the fact that

\[
\begin{align*}
x^o &> x = x^m - y \\
\Rightarrow & \quad \rho^o_y < \rho_y = p^m \\
\Rightarrow & \quad D(\rho^o_y) > D(\rho_y) \\
\Rightarrow & \quad D'(\rho^o_y) \leq D'(\rho_y) \leq 0
\end{align*}
\]

to show that for \( x^o > x^m - y \), \( \partial_x \tilde{\Pi}_a^p(x^m - y, y) > \partial_x \tilde{\Pi}_a^p(x^o, y) \).

Let us first verify the sequence of inequalities above. Based on (45), it is easy to verify the at \( x = x^m - y \),

\[
\rho_y = \frac{\pi^m}{x^m - y} \left( 1 - \frac{y}{D(\rho_y)} \right)
\]

\[
= \frac{\pi^m}{D(\rho_y)} \left( \frac{D(\rho_y) - y}{x^m - y} \right).
\]

If we plug in \( \rho_y = p^m \) to (54), then the left-hand side of (54) simplifies to \( p^m \). Based on expression (52) we know that \( \partial_x \rho_y \), which implies that \( \rho^o_y < \rho_y \). The inequalities \( D(\rho^o_y) > D(\rho_y) \) and \( D'(\rho^o_y) \leq D'(\rho_y) \leq 0 \), follow from Assumption 1 that the demand is decreasing and concave.
Now we get back to showing that \( \partial_x \widetilde{\Pi}_n(x^m - y, y) > \partial_x \widetilde{\Pi}_n(x^o, y) \) for all \( x^o > x^m - y \). Substituting in with (51) and (53) and that \(-a'(x^m - y) > -a'(x^o)\), with some minor algebra we can rewrite \( \partial_x \widetilde{\Pi}_n(x^m - y, y) > \partial_x \widetilde{\Pi}_n(x^o, y) \),
\[
D(p_y^o) \cdot [D'(p_y^o)\pi^m - (D'(p_y^o)p_y^o + D(p_y^o))] x < (x^m)^2 D'(p_y^o)p_y^o.
\]
Since \( p_y^o < p^m \) and demand is decreasing in \( p \), \( D(p_y^o) > x^m \). Thus,
\[
D(p_y^o)^2 D'(p_y^o)p_y^o < (x^m)^2 D'(p_y^o)p_y^o
\]
We estimate the inequality (55) below using (56).
\[
D(p_y^o) \cdot [D'(p_y^o)\pi^m - (D'(p_y^o)p_y^o + D(p_y^o))] x < D(p_y^o)^2 D'(p_y^o)p_y^o,
\]
\[
-(D'(p_y^o)p_y^o + D(p_y^o)) x < D'(p_y^o)(D(p_y^o)p_y^o - \pi^m)
\]
Since \( p_y^o < p^m \), \( D'(p_y^o)p_y^o + D(p_y^o) > 0 \) and \( D'(p_y^o)(D(p_y^o)p_y^o - \pi^m) > 0 \), the final inequality, (57), is satisfied. Finally, if the inequality in (57) is satisfied, then the inequality in (55) must also be satisfied, which implies that for all \( x^o \in (x^m - y, X] \) and \( y \in [0, x^m] \), \( \partial_x \widetilde{\Pi}_n(x^m - y, y) > \partial_x \widetilde{\Pi}_n(x^o, y) \).

**Proof of Theorem 2.** First, we establish that under condition (\( \dagger \)), \((x^*, y^*)\) is a Nash equilibrium of the sequential pricing capacity choice game. Note that since \((x^*, y^*) \in \mathbb{C}\), if \( x \in [0, x^*]\), then \((x, y^*) \in \mathbb{C}\) and \( \Pi^c_n(x, y^*) = \Pi^c_n(x, y^*) \). Also, since \((x^*, y^*)\) is the unique Nash equilibrium of the Cournot game, \( \Pi^c_n(x, y^*) = \Pi^c_n(x^*, y^*) \) for all \( x \in [0, x^*] \). Thus, we are only left to show that there is no profitable deviation \( x^0 \) from \((x^*, y^*)\) such that \( x^0 \in (x^*, X] \).

The following two sequences of inequalities/equalities follow from facts established previously in this section; the fact used for each inequality is placed above the relation. If \( x^o \in (x^m - y^*, X] \) and \( y^* \in [0, x^m] \), then:
\[
\partial_x \widetilde{\Pi}_n(x^o, y^*) \stackrel{(P2)}{<} \partial_x \widetilde{\Pi}_n(x^m - y^*, y^*) \stackrel{(P1)}{=} \partial_x \Pi^c_n(x^m - y^*, y^*) \leq \partial_x \Pi^c_n(x^*, y^*) \stackrel{(E)}{=} 0.
\]

The sequence of inequalities tells us that \( \partial_x \widetilde{\Pi}_n(x^o, y^*) < 0 \) for all \( x^o \in (x^m - y^*, X] \) and \( y^* \in [0, x^m] \), hence \( x^o \notin \widetilde{\Pi}_n^o(y^*) \). Thus, \((x^*, y^*)\) is a Nash equilibrium of the sequential pricing capacity choice game.

Finally, we show that \((x^*, y^*)\) is a Nash equilibrium of the simultaneous pricing capacity choice game. We do this by showing that if \( x^o > x^m - y^* \) and \( y^* \leq x^m \), then \( \Pi^c_n(x^o, y^*) \leq \Pi^c_n(x^o, y^*) \). Since the respective profits are the revenues \( R^p(x^o, y^*) \) and \( R^p(x^o, y^*) \) minus the same cost \( a(x) \), for all \((x^*, y^*) \in \mathbb{C}\) and \( x \in (x^m - y^*, X] \), we can focus on showing that \( R^p(x^o, y^*) \leq \Pi^c_n(x^o, y^*) \). For all \((x, y) \in M^S_a \cup M^L_a \), \( R^p(x, y) \leq \Pi^c_n(x, y) \) if and only if \( p(y, x) \leq \rho^2(y, x) \).
Let us suppose to the contrary that \( \rho(y, x) > \rho^f(y, x) \) and show a contradiction. Based on the fact that \( p \cdot \min\{y, D(p)\} \) is strictly increasing in \( p \) on \([0, p^m]\),

\[
\rho(y, x) \cdot \min\{x, D(\rho(y, x))\} > \rho^f(y, x) \cdot \min\{x, D(\rho^f(y, x))\}
\]

(58)

\[
R^p(x, y) > \tilde{R}^p(x, y)
\]

Since \( \rho(y, x) > \rho^f(y, x) \), we can alternatively write

\[
\tilde{R}^p(x, y) = \pi^m \left( 1 - \frac{y}{D(\rho(y, x))} \right)
\]

> \( \pi^m \left( 1 - \frac{y}{D(\rho(y, x))} \right) \).

The inequality of the second line follows from the Assumption 1, that \( D(p) \) is decreasing in \( p \). Again based on the Assumption 1, we have

\[
R^p(x, y) = \pi^m \int_{\rho(y, x)}^{P(y)} \left( 1 - \frac{y}{D(z)} \right) d\phi^*(z)
\]

< \( \pi^m \left( 1 - \frac{y}{D(\rho(y, x))} \right) \).

Putting the inequalities together we have \( R^p(x, y) < \tilde{R}^p(x, y) \), a contradiction to (58).

Since the respective profits are the revenues \( R^p(x, y) \) and \( \tilde{R}^p(x, y) \) minus the same cost \( a(x) \), for all \( (x^*, y^*) \in \mathcal{C} \) and \( x \in (x^m - y^*, \bar{X}] \), \( \Pi_a^p(x, y^*) \leq \Pi_a^p(x, y^*) < \Pi_a^c(x^*, y^*) \), i.e. \( x \notin r^p_a(y^*) \).

**Proof of Lemma 2.** The proof is separated into two parts. In Part 1 we show that if condition (†) holds, then \( r_a^p(y) = r_a(y) \) for all \( y \in [0, y^*] \). In Part 2 we show that for all \( y \in [y^*, \bar{X}] \) and \( x \in (x^*, \bar{X}] \), \( x \notin r_a^p(y) \). These two parts come together to give the best response characterization in the lemma.

**Part 1:** We show that if (†) and (‡) hold, then \( r_a^p(y) = r_a(y) \) for all \( y \in [0, y^*] \), by first proving the same property for the best response correspondence of the sequential pricing capacity choice game. Given conditions (†) and (‡) hold, (P1) and (P2) together imply for all \( y \in [0, y^*] \) and \( x^o \in [x^m - y^*, \bar{X}] \),

\[
\partial_x \tilde{\Pi}_a^p(x^o, y) \stackrel{(P2)}{=} \partial_x \tilde{\Pi}_a^p(x^m - y, y) \stackrel{(P1)}{=} \partial_x \Pi_a^c(x^m - y, y) \stackrel{(C2)}{=} \partial_x \Pi_a^c(r_a(y), y) \stackrel{(E)}{=} 0.
\]
The sequence of inequalities implies that for all \( y \in [0, y^*] \) and \( x \in \tilde{r}^p_a(y) \), \((x, y) \in \mathbb{C}\), which in turn implies that \( \tilde{r}^p_a(y) = r_a(y) \) for all \( y \in [0, y^*] \). Thus, for all \( x \neq r_a(y) \), \( \tilde{\Pi}_a^p(x, y) < \Pi_a^c(r_a(y), y) \).

Note that by definition \( \Pi_a^p(r_a(y), y) = \Pi_a^c(r_a(y), y) \). Based on the proof of Theorem 2 we know that \( \Pi_a^p(x, y) \leq \tilde{\Pi}_a^p(x, y) \) for all \( (x, y) \in \mathbb{M}_a^S \cup \mathbb{M}_a^L \). Therefore, it must be that for all \( x \neq r_a(y) \), \( \Pi_a^p(x, y) < \Pi_a^c(r_a(y), y) \), which implies that \( r_a^p(y) = r_a(y) \) for all \( y \in [0, y^*] \).

**Part 2:** In this part, we show that for all \( x \in r_a^p(y) \) and \( y \in [y^*, \bar{X}] \), \( x \leq x^* \). Based on Fact 1(b), \( r_a(y) \) is strictly decreasing for all \( y \) such that \( r_a(y) > 0 \). It is immediate that for all \( y \in [y^*, \bar{X}] \) and \( x > x^* \) such that \((x, y) \in \mathbb{C}\), \( \Pi_a^p(x, y) < \Pi_a^c(r_a(y), y) \). Hence, \( x \neq r^p(y) \) for all \( y \in [y^*, \bar{X}] \).

Next we move to showing that for all \( y \in [y^*, \bar{X}] \) and \( x > x^* \) such that \((x, y) \in \mathbb{M}_a^S \cup \mathbb{M}_a^L \), \( x \neq r^p(y) \). By inspection of expressions (19) and (20): for all \((x, y) \in \mathbb{M}_a^S \), \( \rho(x, y) < p^m \), and similarly \( \rho(y, x) < p^m \) for all \((x, y) \in \mathbb{M}_a^L \). The difference in revenue of \( x > x^* \) and \( x^* \) must at most \( p^m(x - x^*) \), while the difference in costs must be at least \( a'(x^*)(x - x^*) \). From this we know that

\[
\Pi_a^p(x, y) \leq \Pi_a^p(x^*, y) - (p^m - a'(x^*)) (x - x^*)
\]

Hence, all such \( x \) cannot be a best response to any such \( y \). ■

**References**


