Cournot outcomes under Bertrand-Edgeworth competition with demand uncertainty

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November 25, 2011

Abstract

We provide new results for two-stage games in which firms make capacity investments when demand is uncertain, then, when demand is realized, compete in prices. We consider games with demand rationing schemes ranging from efficient to proportional rationing. In all cases, there is a subgame perfect equilibrium outcome coinciding with the outcome of the Cournot game with demand uncertainty if and only if (i) the fluctuation in absolute market size is small relative to the cost of capacity, or (ii) uncertainty is such that with high probability the market demand is very large and with the remaining probability the market demand is extremely small. Otherwise, equilibria involve mixed strategies. Further, we show under efficient rationing that condition (i) is sufficient for the unique equilibrium outcome to be an equilibrium outcome of the Cournot game with demand uncertainty.

Keywords Bertrand-Edgeworth duopoly; Demand rationing; Cournot duopoly; Demand uncertainty

JEL Classifications D21 · D43 · L11 · L13

∗I would like to thank Massi DeSantis, Wei-Min Hu, Chris Knittel, Klaus Nehring, Burkhard Schipper and Aric Shafran for their helpful comments. I would like to particularly thank Louis Makowski and Joaquim Silvestre for helping with the development of this manuscript.

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1 Introduction

We analyze competition between firms that must invest in capacity before demand is known. After these capacity investments are sunk, demand is realized and competition commences in prices. The goal of this paper is to understand when capacity investment decisions in such a scenario can be modeled by using market clearing prices à la Cournot. That is, we aim to understand the general conditions under which the conclusion of Kreps and Scheinkman (1983) (hereafter K&S) holds with uncertain demand.¹

Two previous papers have analyzed closely related two-stage games.² As a preliminary step in the analysis of collusive equilibria of a repeated game, Staiger and Wolak (1992) study a two-stage game with linear demand, continuously distributed uncertainty and efficient rationing. They make the incorrect claim that a unique pure symmetric capacity subgame perfect equilibrium always exists.³ This error was pointed out by Reynolds and Wilson (2000) (hereafter R&W) who generalize the two-stage game by allowing a more general class of concave demand functions. R&W’s primary theorem gives necessary and sufficient conditions for the uncertain Cournot outcome to be an equilibrium outcome of the two-stage game. A very interesting insight from R&W is that their model often involves asymmetric capacity investment and mixed strategy pricing. Our specification nests the aforementioned articles including a extremely wide range of demand uncertainty, as well as, a range of residual demand rationing schemes from efficient to proportional rationing.⁴ We provide more general results that are intuitively similar to R&W’s claims, and further, provide results pertaining to uniqueness of equilibrium.

In a preliminary step of our analysis, we characterize the equilibria of a Cournot game with demand uncertainty. The Cournot game with demand uncertainty may possess many conditions for the uncertain Cournot outcome to be an equilibrium outcome of the two-stage game. A very interesting insight from R&W is that their model often involves asymmetric capacity investment and mixed strategy pricing. Our specification nests the aforementioned articles including a extremely wide range of demand uncertainty, as well as, a range of residual demand rationing schemes from efficient to proportional rationing.⁴ We provide more general results that are intuitively similar to R&W’s claims, and further, provide results pertaining to uniqueness of equilibrium.

¹There is a significant literature, not directly related to our purpose, taking the analysis of the two-stage games in many interesting directions. Some examples are: Deneckere and Kovenock (1992) (Capacity constrained with sequential pricing); Allen (1993) and Allen et al. (2000) (sequential capacity choice game with simultaneous pricing); Deneckere and Kovenock (1996) (asymmetric costs of production up to capacity); Maggi (1996) and Boccard and Wauthy (2000,2004) (costly adjustment of capacity in the second stage).

²In a related paper, Hviid (1991) studies Betrand-Edgeworth pricing games where demand is still uncertain when prices are chosen. Fabra and de Frutos (2011) also analyze a game two-stage game closely related to ours. They assume demand is composed of a mass of consumers who each demand one unit if the price is less than or equal to one and zero otherwise. The firms are uncertain about the mass of consumers that will be at the market. One of the special features of these rectangular demand functions is that all rationing rules are the same and are efficient. Since we only consider downward sloping demand, their game is a limiting case of our model, just outside of our specification.

³The incorrect claim is actually fairly innocuous to their primary purpose, the study of collusive pricing. For a discussion of this point see Reynolds and Wilson (2000), Section 4.

⁴K&S (1983) results only pertain to the case of efficient rationing of residual demand. Davidson and Deneckere (1986) make a compelling argument for the use proportional rationing and show the Cournot outcome is not always the equilibrium outcome of the two-stage game with proportional rationing. Lepore (2009) derives precise conditions when the Cournot outcome is an equilibrium of the two-stage game with proportional rationing.
equilibria (hereafter labeled $UCE$), which are all symmetric. We denote by an uncertain Cournot outcome ($UCO$), the prices and capacities (quantities) of a UCE.\footnote{Throughout the paper the term capacity and quantity will be used as synonyms.}

The analysis of the two-stage game is restricted to subgame perfect equilibria and hence, we analyze a game where firms choose only capacities and expected revenue is determined by the Nash equilibrium of the pricing subgame. We denote this game the capacity choice game.

Our first result regarding the capacity choice game is that there exists an equilibrium with a UCO if and only if with probability one, each firm’s best response to its rival choosing the largest uncertain Cournot capacity leads to pure strategy pricing. There are two scenarios where this condition is satisfied: (i) the range of uncertainty in demand is small and the cost of capacity is relatively high, or (ii) there is high probability of large demand states, low probability of states with very limited absolute demand and the cost of capacity is relatively low.

A sufficient condition for the UCO to be the unique equilibrium outcome of the capacity choice game is that neither firm will ever choose to be the weakly larger firm when mixed strategy pricing can occur with positive probability. Under efficient rationing, this condition is satisfied if the largest UCE capacities lead to Cournot pricing with probability one, which is true for scenario (i) above.

After presenting the general results, we restrict uncertainty to be continuously distributed and compare our results with R&W’s. This uncovers an error in the characterization of R&W. The characterization we find is slightly different than R&W’s, but similar in economic content. We provide additional assumptions, stronger than R&W’s, which recover their exact characterization. Further, under efficient rationing we show that if an equilibrium of the capacity choice game has a UCO, then it is the unique equilibrium of the capacity choice game.

The structure of the rest of the paper goes as follows. The basic assumptions and definitions of the model are outlined in Section 2. In Section 3, the set of UCE is characterized. Section 4 is a synthetic characterization of the Nash equilibria of the pricing subgames for all rationing schemes. The general results are stated in Section 5. In Section 6, we apply our general results to the special case of uncertainty following a continuous distribution, and Section 7 concludes the paper.

## 2 Model Basics

The model has two firms, 1 and 2, that choose capacities denoted $x$ and $y$, respectively. The parameter $\theta$ is the market demand level. It is a random variable with a particular
distribution $\mu$ on the compact space $\Theta \subset \mathbb{R}_+^n$. The market inverse demand is the function $P(q, \theta) : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ and the market demand is the function $D(p, \theta) : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$. We state the formal assumptions:

**Assumption 1** For each $\theta \in \Theta$, there exists a quantity $X(\theta)$ such that $\forall q \in [0, X(\theta))$, $P(q, \theta) \in (0, \infty)$ and $\forall q \geq X(\theta)$, $P(q, \theta) = 0$. On $(0, X(\theta))$, $P(q, \theta)$ is twice-continuously differentiable, strictly decreasing and concave in $q$.

**Assumption 2** For all $\theta, \theta' \in \Theta$ such that $\theta \neq \theta'$, there exists $q \in \mathbb{R}_+$ such that $P(q, \theta) \neq P(q, \theta')$.

The second assumption imposes minimal restriction on the nature of demand uncertainty. For the set of parameters $\Theta$, inverse demand functions are allowed to cross, become more or less concave, and move discontinuously in the parameter $\theta$. For notational convenience, we will often write the partial derivative with respect to the first argument as $\partial_q P(x, \theta) = \frac{\partial P(q, \theta)}{\partial q}|_{q=x}$.

**Assumption 3** Both firms face the same capacity cost function $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$, which is twice-continuously differentiable, nondecreasing, convex, and satisfies $a(0) = 0$. The cost function is such that $0 < a'(0) < E[\partial_q P(0, \theta) + P(0, \theta)]$.

Define $\bar{X} = \max X(\theta)$. For each state inverse demand function, as $q$ goes toward the upper bound $X(\theta)$, we assume the derivative of the inverse demand is bounded.

**Assumption 4** For all $\theta \in \Theta$, there exists a finite $\gamma(\theta) \in \mathbb{R}_+$ such that

$$\lim_{x \uparrow X(\theta)} \partial_q P(x, \theta) \geq \gamma(\theta).$$

The two-stage game has the following timing. At the beginning of the game, the demand parameter $\theta$ is uncertain and each firm simultaneously and independently chooses its capacity level. Both firms observe the realized demand parameter $\theta$ and the other firm’s capacity, then choose simultaneously and independently their prices.

We restrict the analysis of the two-stage game to subgame perfect equilibria. Hence, we study a capacity choice game where expected revenues are taken as deterministic functions of capacities. The expected revenue functions are constructed from the Nash equilibria of the pricing subgames.

In addition, two auxiliary games – the Cournot game and the uncertain Cournot game – are used to characterize the equilibria of the capacity choice game.
The first auxiliary game is the standard Cournot game for a given demand parameter \( \theta \in \Theta \) and zero cost. It is well known that under assumptions 1-4, given a fixed \( \theta \in \Theta \), there exists a unique Cournot equilibrium, which is symmetric. Denoted by \( \hat{q}(\theta) \) a firm’s equilibrium quantity (capacity).

The second auxiliary game, the uncertain Cournot game, has the following timing. At the beginning of the game the demand parameter \( \theta \) is uncertain and each firm chooses its capacity level independently and simultaneously. Then the demand parameter is realized and price is determined by market clearing such that \( p_1 = p_2 = P(q_1 + q_2, \theta) \). Section 3 is devoted to characterizing the equilibria of the uncertain Cournot game.

### 3 Uncertain Cournot Game

The purpose of this section is to establish the character of the set of UCE, which is the basis for understanding the extension of the K&S result to demand uncertainty.

Under assumptions 1-4, the uncertain Cournot profit function is not necessarily quasi-concave. In figure 1 we plot the expected revenue function for the case of a high state/low state binomial distribution. Even for this simple case, variation in the quantity intercepts \( (X(\theta)) \) leads to non quasi-concave expected revenues.

![Figure 1: An example of a non quasi-concave expected revenue](image-url)

For this reason, it is not immediately obvious that a pure strategy Nash equilibrium exists. Based on cost symmetry and the fact that the best response functions can only jump upward, existence of a pure strategy equilibrium follows from Roberts and Sonnenschein (1976).\(^7\)

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\(^6\)See Section 3 in Kreps and Scheinkman (1983).

\(^7\)Roberts and Sonnenschein (1976) existence theorem is based on a subcase of Tarski's fixed point theorem. This relates to the more recent lattice theoretic techniques to prove of existence of pure strategy equilibrium, see Vives (1999) for an insightful discussion.
In the following theorem we prove that all pure strategy UCE quantities (capacities) are symmetric and must satisfy a right-hand derivative condition. These two facts are extremely useful in the configuration of our primary results. Denote by $q^*$ an equilibrium quantity of the uncertain Cournot game and denote the largest of all such symmetric UCE capacities by $\overline{q}$.

In order to establish notation, the *uncertain Cournot expected profit* is

$$\Pi_c(x, y) = E[P(x + y, \theta)x] - a(x), \quad (1)$$

and an *uncertain Cournot game best response* correspondence is

$$\beta_c(y) = \arg \max_{x \in [0, X]} \Pi_c(x, y). \quad (2)$$

**Theorem 1** Every pure strategy UCE is symmetric and satisfies the condition

$$E[\partial_q P(2q^*, \theta)q^* + P(2q^*, \theta)] - a'(q^*) = 0,$$

where $U(q) = \{\theta \in \Theta : q < D(0, \theta)\}$.\(^8\)

All proofs are provided in the appendix.

4 The Pricing Subgames

Before we move to the characterization of the Nash equilibria of pricing subgames, we formally address the way in which demand is rationed.

4.1 Rationing Schemes

More specifically, the demand served by firm $i$ is:

$$D_i^c(p_1, p_2, \theta) = \begin{cases} \min \{x, D(p_1, \theta)\} & \text{if } p_i < p_j \\ \min \left\{ x, \max \left\{ \frac{D(p_1, \theta)}{2}, D(p_i, \theta) - y \right\} \right\} & \text{if } p_i = p_j \\ \min \left\{ x, \max \{0, d_i^c(p_1, p_2, \theta)\} \right\} & \text{if } p_i > p_j \end{cases} \quad (3)$$

\(^8\)At any $q = D(0, \theta)$, the revenue function for state $\theta$ is not differentiable. This is apparent by the kink in the expected revenue function of figure 1. The right-hand derivative of the revenue function for state $\theta$ must be zero at this point. Thus, our restriction to the set $U(2q)$ removes these non-differentiable points from the expression; giving the right-hand derivative at $q$.  

[531x800]
where \( r \in \{e, p\} \), “e” represents “efficient rationing” and “p” represents “proportional rationing.” The residual demand \( d_i^r(p_1, p_2, \theta) \) is the only term that varies with \( r \). The two residual demands for \( r = e \) and \( r = p \) are:

\[
d_e^r(p_i, p_j, \theta) = D(p_i, \theta) - y, \\
d_p^r(p_i, p_j, \theta) = D(p_i, \theta) \left(1 - \frac{y}{D(p_j, \theta)}\right).
\]

It is also necessary to define a series of demand rationing schemes that are between efficient and proportional rationing. The set of rationing schemes \( \Gamma \) is such that, for all \((p_1, p_2)\), \( d_e^r(p_1, p_2, \theta) \in [D(p_i, \theta) - y, D(p_i, \theta)(1 - y/D(p_j, \theta))] \) and \( d_p^r(p_1, p_2, \theta) \) is continuous and weakly decreasing in both the prices.

In what follows, we will specify results that pertain to the all rationing schemes \( \Gamma \), as opposed to results that pertain to a particular scheme.

### 4.2 Equilibria of Pricing Subgames

There is a pricing subgame for each \( \theta \in \Theta \). Here we fix the two firms’ capacity choices at arbitrary values \( x \) and \( y \) and examine the pricing subgame given any demand realization.

We denote equilibrium mixed strategies of the pricing subgame of demand parameter \( \theta \) by \((\Phi_1^r, \Phi_2^r)\). The characterization below follows from the characterization shown in Lepore (2009), which is based on results from K&S, Davidson and Deneckere (1986) and Deneckere and Kovenock (1992). In order to delineate pricing regions, we define the Cournot best response function for demand parameter \( \theta \) for a firm with zero cost

\[
r(y, \theta) = \arg \max_{x \in [0, X(\theta)]} P(x + y, \theta)x.
\]

Denote by \( x^m(\theta) \) and \( p^m(\theta) \), the zero cost monopoly capacity (quantity) and price, respectively, for demand parameter \( \theta \).

It is useful to classify the three regions of equilibrium pricing based on the demand parameter \( \theta \). For either rationing rule, the unique equilibrium is cut-throat Bertrand pricing for parameter \( \theta \) if \( \min\{x, y\} \geq X(\theta) \). Formally, the set of Bertrand pricing parameters, for any \( x \) and \( y \), is

\[
\mathbb{B}(x, y) = \{\theta \in \Theta \mid \min\{x, y\} \geq X(\theta)\}.
\]

The region of Cournot pricing depends on the rationing rule. The set of demand parameters with Cournot pricing under proportional rationing is \( \mathbb{C}^p(x, y) \), under efficient rationing it is

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9Existence of mixed strategy Nash equilibria in the capacity constrained pricing games is based on results in Dasgupta and Maskin (1986a&b) and Maskin (1986).
\( \mathcal{C}^e(x, y) \). Formally,

\[
\mathcal{C}^p(x, y) = \{ \theta \in \Theta \mid x + y \leq x^m(\theta) \}, \\
\mathcal{C}^e(x, y) = \{ \theta \in \Theta \mid x \leq r(y, \theta) \text{ and } y \leq r(x, \theta) \}.
\]

Notice that \( \mathcal{C}^p(x, y) \subset \mathcal{C}^e(x, y) \).

For all rationing schemes \( r \in \Gamma \), the set of capacities such that the subgame equilibrium has Cournot pricing is \( \mathcal{C}^r(x, y) \). The set \( \mathcal{C}^r(x, y) \) is nonempty and \( \mathcal{C}^p(x, y) \subset \mathcal{C}^r(x, y) \subset \mathcal{C}^e(x, y) \). The following lemma establishes this formally.

**Lemma 1** For all \( r \in \Gamma \), \( \mathcal{C}^r(x, y) \subseteq \mathcal{C}^e(x, y) \) and \( \mathcal{C}^p(x, y) \subseteq \mathcal{C}^r(x, y) \).

Under any rule \( r \in \Gamma \), for any other capacity combinations the existence of equilibria is only guaranteed in mixed strategy pricing. We define the mixed strategy pricing regions below.

\[
\mathcal{M}^r_i(x, y) = \{ \theta \in \Theta \mid \theta \notin \mathcal{C}^r(x, y), y < X(\theta) \text{ and } x \geq \gamma \}, \\
\mathcal{m}^r_i(x, y) = \{ \theta \in \Theta \mid \theta \notin \mathcal{C}^r(x, y), x < X(\theta) \text{ and } x < \gamma \}.
\]

Define \( \mathcal{M}^r(x, y) = \mathcal{m}^r_i(x, y) \cup \mathcal{M}^r_i(x, y) \), the set of all mixed capacities for the parameter \( \theta \).

We now focus on a characterization of the equilibrium expected revenue the cases \( r \in \{e, p\} \). We can be more specific about the the Nash equilibrium expected revenue of firm \( i \) is for the cases \( r \in \{e, p\} \),

\[
R^r(x, y, \theta) = \begin{cases} 
P(x + y, \theta)x & \text{if } \theta \in \mathcal{C}^r(x, y) \\
v^r(y, \theta) & \text{if } \theta \in \mathcal{M}^r_i(x, y) \\
p^r(x, y, \theta)x & \text{if } \theta \in \mathcal{m}^r_i(x, y) \\
0 & \text{if } \theta \in \mathcal{B}(x, y),
\end{cases}
\]

where,

\[
v^e(y, \theta) = P(r(y, \theta) + y, \theta)r(y, \theta), \\
v^p(y, \theta) = p^m(\theta)x^m(\theta) \int_{P(y, \theta)}^{y} \left( 1 - \frac{y}{D(z, \theta)} \right) d\Phi_2(z).
\]

The term \( p^r(x, y, \theta) \) is the lowest price in the support of equilibrium pricing, it is a piecewise function defined in what follows. If \( \theta \in \mathcal{M}^r(x, y) \), then the expected revenue of each firm is determined by the lowest price in the support of mixed strategy of the smaller firms. There are two possible lowest prices of the support. First

\[
p^r(x, y, \theta) = \min \{ p \in \mathbb{R}_+ \mid py = v^r(x, \theta) \}.
\]
The function \( r \) is defined for \( x > 0 \) \( \forall x \in (0, X) \). The second possible price applies when the opposing firm is larger,

\[
\phi^r(x, \theta) = \min \{ p \in \mathbb{R}_+ \mid pD(p, \theta) = v^r(x, \theta) \}.
\]

The function \( \phi^r \) is defined for \( x \geq 0 \), and is twice-continuously differentiable \( \forall x \in (0, X) \). The actual price is the maximum of the two prices in (4) and (5)

\[
p^r(x, y, \theta) = \max \{ \rho^r(x, y, \theta), \phi^r(x, \theta) \}.
\]

For the cases \( r \in \{ e, p \} \), the capacity choice expected profit given Nash equilibrium pricing is defined as

\[
\Pi^r_a(x, y) = E [R^r(x, y, \theta)] - a(x).
\]

Denote the capacity choice best response by

\[
\beta^r_a(y) = \max_{x \in [0, X]} \Pi^r_a(x, y).
\]

Note that, for \( r \in \{ e, p \} \), the function \( \Pi^r_a(x, y) \) is bounded and continuous for all \((x, y) \in \mathbb{R}_+^2\). Hence, for any \( y \in [0, X] \), it attains a maximum on \( x \in [0, X] \). A continuous function on a compact rectangle in \( \mathbb{R}_+^2 \) is uniformly continuous on that rectangle; and, in turn, a uniformly continuous function on a compact space has a closed set of maximizers. Thus, the set \( \beta^r_a(y) \) is closed for all \( y \in [0, X] \) i.e., for all \( y \in [0, X] \) and for any sequence \( \{y^n\} \) that converges to \( y \), we have that \( \lim_{n \to \infty} \beta^r_a(y^n) \in \beta^r_a(y) \).

For the general case of \( r \in \Gamma \), it is not clear that a unique equilibrium expected revenue exists for each pair of capacities \((x, y) \in \mathbb{M}^r(x, y)\). We define some additional notation to help understand the expected revenue of all equilibria for all \( r \in \Gamma \).

Denote the index \( \alpha(\theta) \) for an arbitrary equilibrium expected revenue \( R^r(x, y, \theta : \alpha(x, y, \theta)) \), given capacities \((x, y) \) and the state \( \theta \). The set of all indexes for different equilibria given \((x, y) \) and the state \( \theta \) is denoted by \( \mathcal{A}(x, y, \theta) \). Denote a possible expected profit of firm \( a \) by \( \Pi^r_a(x, y : \alpha) = E [R^r(x, y, \theta : \alpha(x, y, \theta))] - a(x) \) for all \( \alpha \in \mathcal{A}(y) = \prod_{\theta \in \Theta} \mathcal{A}(x, y, \theta) \). Denote the best response given a certain set of equilibrium expected revenues of the pricing subgame \( \alpha \in \mathcal{A}(y) \),

\[
\beta^r_a(y : \alpha) = \max_{x \in [0, X]} \Pi^r_a(x, y : \alpha).
\]

Finally, define the set of all possible best responses to \( y \), for all \( \alpha \in \mathcal{A}(y) \),

\[
\mathcal{B}^r_a(y) = \bigcup_{\alpha \in \mathcal{A}(y)} \beta^r_a(y : \alpha).
\]

In the following preliminary result, we establish that for any rationing \( r \in \Gamma \) all subgame perfect equilibrium expected profit are weakly greater than the expected subgame perfect
equilibrium expected profit with efficient rationing.

Lemma 2 For all \( r \in \Gamma \), \( \Pi^r_\alpha(x, y : \alpha) \geq \Pi^r_\alpha(x, y) \) for all \( (x, y) \in [0, \bar{X}] \times [0, \bar{X}] \) and \( \alpha \in A(y) \).

5 The Capacity Choice Game

We first show the necessary and sufficient conditions for a UCO to be an equilibrium outcome of the capacity choice game. Then we provide a sufficient condition for the unique pure strategy equilibria of each capacity choice game to have a UCO.\(^\text{10}\)

5.1 Existence

Regardless of rationing rule, a variation of the same condition defines the existence of equilibrium with a UCO. There is a capacity choice equilibria with a UCO if and only if there is a best response to \( \bar{q} \) that is not both, larger than \( \bar{q} \) and leads to mixed strategy pricing with positive probability. It also turns out that only the largest capacity UCO can be an equilibrium of the capacity choice game. These facts are formalized in the theorem below.

Theorem 2 Consider the capacity choice game defined above by assumptions 1 – 4.

1. For any rationing rule \( r \in \Gamma \), the only equilibrium of the capacity choice game that can have a UCO is \( x = y = \bar{q} \). Further, if rationing is efficient, then \( x = y = \bar{q} \) are the only symmetric capacities that can be an equilibrium.

2. For any rationing rule \( r \in \Gamma \), the UCE capacities, \( x = y = \bar{q} \), are an equilibrium of the capacity choice game if and only if

\[
\exists x \in B^r_\alpha(\bar{q}) \text{such that } \mu(M^r(x, \bar{q})) = 0. \tag{7}
\]

We provide an example to give some basic context to statements of the theorem. In the example, Condition 7 is violated and there is no equilibrium with the UCO. After the example, we move to a sketch of the proof of Theorem 2.

Example 1 (No UCO) This example is constructed to show that when Condition 7 is not satisfied, \( \bar{q} \) is not an equilibrium. Assume that demand is rationed according to the efficient rationing rule. Define the inverse demand function \( P(q, \theta) = \max\{\theta - q, 0\} \) where \( \theta \in \{8, 18\} \).

\(^{10}\) As a side note, an equilibrium of the capacity choice game in mixed strategies always exists. Existence follows immediately form Glicksberg (1952) since the profit functions are continuous in both firms capacities.
The probability that \( \theta = 8 \) is \( \mu(8) = 1/2 \) and that \( \theta = 18 \) is \( \mu(18) = 1/2 \). The marginal cost of capacity is the constant 1. The unique UCE is the symmetric capacity \( \bar{q} = 4 \). The capacity choice game best response of firm 1 to firm 2 playing \( y = 4 \) is the capacity \( x = 6 \). Thus, \( x = y = 4 \) is not an equilibrium of the capacity choice game. For the demand state \( \theta = 8 \) the capacities \((6, 4)\) result in mixed strategy pricing (this is because \( r(4, \theta = 8) = 2 < 6 \) and \( \min\{4, 6\} < X(\theta = 8) = 8 \)). Therefore \( \mu(\mathcal{M}^c(6, 4)) = 1/2 > 0 \), a violation of Condition 7.

We plot the firms’ best response correspondences in Figure 2. Notice that the discontinuity in the best response correspondences prevents the crossing at any symmetric capacities. There are two asymmetric pure capacity equilibrium with one firm playing 4.86 and the other playing 5.57. In demand state \( \theta = 18 \), equilibrium pricing is market clearing: \( p_1 = p_2 = 7.57 \), while in demand state \( \theta = 8 \), equilibrium pricing is in mixed strategies.

![Figure 2: The best response correspondence of firms when Condition 7 is violated.](image)

The outline of the proof of Theorem 2 goes as follows. We begin the proof of part 1 addressing only the efficient rationing case and establishing two preliminary lemmas. In the first lemma, we establish that there can only be one symmetric pure capacity equilibrium, labeled the candidate. Recall Theorem 1, which shows that all UCE must be symmetric. Thus, only if the candidate is an equilibrium, can the outcome of an equilibrium of the two-stage game coincide with that of an analogue uncertain Cournot equilibrium.

The candidate corresponds with the pure symmetric capacity that Staiger and Wolak (1992) claim is always an equilibrium. We show that the candidate is uniquely defined by the right-hand derivative of the expected profit function of each firm equating to zero. We denote by \( \hat{x} \) the candidate capacity, which is defined by the equality

\[
E[\partial_q P(2\hat{x}, \theta)\hat{x} + P(2\hat{x}, \theta) | C^e(\hat{x}, \hat{x})] - a'(\hat{x}) = 0. \tag{SC}
\]
A second lemma shows that the candidate can only be equal to the largest UCE capacity. The proof is based on showing that condition (SC) is only satisfied by symmetric capacities weakly larger than all capacities that satisfy (CE). Putting the two lemmas together, we show that the candidate can only be an equilibrium if it equals the largest UCE capacity.

Considering all rationing schemes \( r \in \Gamma \), the proof of part 1 of the theorem is based on showing that only \( x = y = \bar{q} \) can be an equilibrium of the capacity choice game and lead to pure strategy pricing with probability one. This is because both conditions (SC) with \( r \in \Gamma \), and (CE) must hold for any symmetric equilibria with probability one of pure strategy pricing. Thus, we can restrict the analysis to \( \bar{q} \) as an equilibrium of the capacity choice game for all rating schemes \( r \in \Gamma \).

Based on part 1, the arguments for necessity and sufficiency in part 2 of the theorem are straightforward. Necessity is essentially trivial. As an equilibrium with no mixed pricing, the largest UCE capacity is a best response to the rival firm choosing the largest UCE capacity.

Sufficiency is shown based on applying the fact that \( \bar{q} \) is a UCE capacity. Each firm must have a best response to \( \bar{q} \) which leads to only Bertrand and Cournot pricing with positive probability. The profit at the best response will be the same in both the uncertain Cournot game and the capacity choice game, which implies that, if \( \bar{q} \) is a UCE, then it must also be an equilibrium of the capacity choice game.

### 5.2 Uniqueness

In this section we focus exclusively on efficient rationing. Before proceeding to the general analysis, we provide an example with multiple equilibria to get a basic understanding of the circumstances when a UCO is not the only equilibrium.

**Example 2 (Multiple equilibria)** The following example has three equilibria, one with a UCO. The market inverse demand function is:

\[
P(q, \theta) = \begin{cases} 
\max\{3 - q, 0\} & \theta \in [0, 0.994] \\
\max\{30 - q, 0\} & \theta \in [0.994, 1]
\end{cases}
\]

where \( \theta \in [0, 1] \). The probability of any realization of \( \theta \) is uniformly distributed on \([0, 1]\). The marginal cost of capacity is the constant 0.001. The largest UCE symmetric quantity is \( \bar{q} = 179/18 \approx 9.94 \). Notice that \( 3 \in \mathbb{B}(9.94, 9.94) \) and \( 30 \in \mathbb{C}(9.94, 9.94) \). We plot the firms best response correspondences in Figure 3. There are three pure capacity equilibrium: two asymmetric equilibria with one firm playing 1.03 and the other playing 14.4, and the UCO \( x = y = \bar{q} \approx 9.94 \).

In the above example, a positive measure of demand states are in the Bertrand pricing region at the largest UCE quantities (\( \mu(\mathbb{B}(\bar{q}, \bar{q})) > 0 \)). If this is not the case, then the
unique equilibrium of the capacity choice game is $x = y = \bar{q}$. This is stated formally in the theorem below.

**Theorem 3** Consider the capacity choice game defined above by assumptions 1–4. For $r = e$, if $\mu(C^e(\bar{q}, \bar{q})) = 1$, then $x = y = \bar{q}$ is the unique Nash equilibrium in pure strategies.

The first part of the proof is based on an extension of the 1st Judo Principle shown in Lepore (2009). The concept of judo economics was introduced by Gelman and Salop (1983) in a model of entry. Gelman and Salop show that a monopolist of unlimited size will be unable to deter a single potential entrant if the entrant can credibly restrict its capacity. As a result the entrant will choose to be ‘small’ and price such that the monopolist will not find it profitable to undercut. This strategy is what Gelman and Salop call “judo economics", because the incumbents large size is being used against itself.

The judo intuition can also be applied to the analysis of the two-stage game. Analyzing a model of two-stage competition with no demand uncertainty, Lepore (2009) identifies the 1st Judo Principle, which is the basic property that no firm will ever choose to be the victim of judo. That is, no firm will ever find it optimal to be (weakly) larger than its rival if the ensuing Nash equilibrium pricing subgame will be in the mixed pricing region. Instead, the firm will prefer to be smaller and avoid being attacked by the judo strategy. The intuition behind this principle is simple. If firm $a$ is larger than firm $b$ and the Nash equilibrium of the pricing subgame involves mixed strategies, the larger firm makes the same expected revenue for all capacities greater than or equal to the zero cost Cournot best response to its rival. So, because capacity is costly, the larger firm increases its expected profit by reducing its
capacity to exactly the zero cost Cournot best response to its rival, which moves the pricing subgame from the mixed strategy region to into the pure strategy market clearing pricing region.

In the uncertain demand model with efficient rationing the extension of 1st Judo principle is: Given that the largest UCE capacities lead to non-zero market clearing prices with probability 1, no firm will ever choose to have be the victim of judo with positive probability. We can restate this result more formally as: \( \mu (C^*(\overline{q}, \overline{q})) = 1 \), which implies that no firm will ever want to be the weakly larger firms when there is positive probability that the pricing is in mixed strategies.

The rest of the proof goes as follows. The condition \( \mu (C^*(\overline{q}, \overline{q})) = 1 \) implies that (7) holds at \( x = y = \overline{q} \). Hence, based on Theorem 1, \( x = y = \overline{q} \) is a Nash equilibrium of the capacity choice game. There cannot be an equilibrium with positive probability of mixed strategies, because one firm must be weakly larger with positive probability. Thus, equilibria can only be such that Cournot and Bertrand pricing are probable outcomes. The only equilibrium of this form is \( x = y = \overline{q} \). Therefore, \( x = y = \overline{q} \) is the unique Nash equilibrium in pure strategies of the capacity choice game.

In the following section we restrict the model to get a more intuitive characterization and compare the results to R&W.

### 6 A Restricted Characterization

In this section we provide special characterizations with more restricted assumptions. This treatment is closely related to the analysis done in R&W. R&W assume the distribution of uncertainty is continuous with full support on the parameter space and that demand is rationed using the efficient rule.\(^{11}\) In R&W’s Theorem 1 they claim that with efficient rationing, \( x = y = q^* \) is a Nash equilibrium of the capacity choice game if and only if \( \hat{q}(\hat{\theta}) \geq q^* \). It turns out this condition does not characterize \( x = y = q^* \) as a Nash equilibrium of the capacity choice game. In Section 3 we hinted at the fact that multiple UCE can exist under our assumptions. The assumptions of R&W do not preclude this possibility. This provides an immediate problem with their result (since \( q^* \) is not well defined). Even when the UCE is unique, the conditions of R&W are neither necessary nor sufficient. In the following example, with a unique UCE, we show that the conditions of R&W are not necessary for the existence of a UCO.

**Example 3 (Counter example to R&W )** Let us reconsider a slightly different version

\(^{11}\)R&W also assume that marginal cost is constant and a larger demand state indexes a larger inverse demand for all fixed quantities: \( \Theta = [\underline{\theta}, \overline{\theta}] \), and for all \( \theta, \theta' \in [\underline{\theta}, \overline{\theta}] \) such that \( \theta > \theta', P(q, \theta) > P(q, \theta') \) for all \( q \in \mathbb{R}_+ \).
of Example 2, which fits the assumption of R&W. Define the market inverse demand:

\[
P(q, \theta) = \begin{cases} 
\max\{3 + \epsilon\theta - q, 0\} & \theta \in [0.0994] \\
\max\{30 + \epsilon\theta - q, 0\} & \theta \in [0.994, 1] 
\end{cases}
\]

where \( \epsilon = 1 \times 10^{-200} \). Since \( \theta = 0 \), \( \tilde{q}(0) = 1/3 \) and the unique UCE of the game is \( q^* \approx 9.94 \). Just as in example 2, \( x = y = q^* \approx 9.94 \) is an equilibrium of the capacity choice game. But the condition of R&W that \( \tilde{q}(\theta) \geq q^* \) is clearly violated since \( \tilde{q}(0) = 1/3 < 9.94 = q^* \).

In what follows, we add assumptions and incrementally arrive at more restrictive characterizations. With three additional assumptions, we are able to recover exactly R&W’s characterization. Further, we show on this restricted domain that R&W’s conditions also characterize uniqueness of equilibrium.

Let us begin with a restriction that the inverse demand is continuous in the uncertainty parameter and the distribution of uncertainty has full support on a compact interval of values.

**Assumption 5** \( \forall q \in (0, X(\theta)), P(q, \theta) \) is continuous in \( \theta \) on \( [\overline{\theta}, \overline{\theta}] \), and that \( \mu \) is positive with continuous support on \( [\overline{\theta}, \overline{\theta}] \), where \( 0 \leq \overline{\theta} < \overline{\theta} < \infty \).

R&W assume the distribution of uncertainty is continuous with full support on the parameter space. We have added the assumption that the inverse demand changes continuously in the demand parameter. With this assumption we get a characterization for efficient rationing with a similar flavor to R&W’s. Further, our characterization is stronger as it provides a necessary and sufficient condition for the Cournot outcome to be the unique pure strategy equilibrium outcome under efficient rationing.

**Corollary 1** Under assumptions 1-5:

For \( r = e \):

(e.1) If \( \min \tilde{q}(\theta) \geq \tilde{q} \), then \( x = y = \tilde{q} \) is the unique Nash equilibrium in pure strategies.

(e.2) If \( \min \tilde{q}(\theta) < \tilde{q} \), then \( x = y = \overline{q} \) is not a Nash equilibrium.

For \( r = p \):

(p.1) If \( \min x^m(\theta) \geq \tilde{q} \), then \( x = y = \overline{q} \) is a Nash equilibrium in pure strategies.

(p.2) If \( \min x^m(\theta) < \tilde{q} \), then \( x = y = \overline{q} \) is not a Nash equilibrium.
Based on Assumption 5, \( \min \hat{q}(\theta) \geq \bar{q} \) and \( \min \hat{q}(\theta) < \bar{q} \) imply \( \mu(M_i^e(q, \bar{q})) = 0 \) and \( \mu(M_i^e(q, \bar{q})) > 0 \), respectively. Hence, the proof of existence is an immediate consequence of Theorem 2, while uniqueness is an immediate consequence of Theorem 3.

To recover the exact characterization of R&W we must add two more assumptions. The addition of the following condition on the inverse demand and nature of uncertainty is sufficient to guarantee a unique UCE.

**Assumption 6** \( \forall q \in (0, X(\theta)), P(q, \theta) \) and \( \mu \) are continuously differentiable in \( \theta \) on \( [\underline{\theta}, \overline{\theta}] \).

**Proposition 1** Under assumptions 1-6, the uncertain Cournot game has a unique equilibrium.

To recover the exact characterization of R&W we add the final assumption.

**Assumption 7** \( \forall (\theta, \theta') \in [\underline{\theta}, \overline{\theta}] \times [\underline{\theta}, \overline{\theta}] \) and \( \forall q \in (0, X(\theta)) \), \( \partial_q P(q, \theta') \geq \partial_q P(q, \theta) \) if \( \theta' > \theta \).

As shown in Proposition 1, the addition of Assumption 6 guarantees uniqueness of UCE. While, Assumption 7 ensures that \( \min \hat{q}(\theta) = \hat{q}(\theta) \). Hence, the two conditions \( \min \hat{q}(\theta) \geq \bar{q} \) and \( \min \hat{q}(\theta) < \bar{q} \) can be re-written as \( \hat{q}(\theta) \geq q^* \) and \( \hat{q}(\theta) < q^* \), respectively. This leads to the following result where R&W’s conditions are necessary and sufficient for the existence of the UCO in the capacity choice game. Based on our new results, the condition for the existence of a UCO is also necessary and sufficient for the UCO to be the unique equilibrium outcome of the capacity choice game.

**Corollary 2** Under assumptions 1-7 and for \( r = e \):  
1. If \( \hat{q}(\theta) \geq q^* \), then \( x = y = q^* \) is the unique Nash equilibrium in pure strategies.  
2. If \( \hat{q}(\theta) < q^* \), then \( x = y = q^* \) is not a Nash equilibrium.

## 7 Conclusion

We have characterized the conditions under which a UCE outcome is an equilibrium outcome of a fairly general class of two-stage games with a broad class of rationing schemes ranging from efficient to proportional rationing. Further, we have shown that under efficient rationing the condition for existence of a UCO is also necessary and sufficient for uniqueness of equilibrium. Thus, we can say that the K&S result is robust to demand uncertainty, if the range of demand uncertainty about the number of the quantity demanded is low or the marginal cost of capacity is high enough. We provide a context for when the uncertain Cournot game is an adequate reduced form for the two-stage capacity pricing game with demand “noise.” We leave for future research the exploration of further equilibrium character when no uncertain Cournot outcome is an equilibrium outcome of the two-stage game.
8 Appendix

Proof of Theorem 1. The profit function $\Pi_i^c(q_i, q_j) = E[P(q_1 + q_2, \theta)q_i] - a(q_i)$, for $i \neq j$ and $i \in \{1, 2\}$, is not necessarily differentiable at all $q_i$ in $(0, \bar{X} - q_j)$ because at each quantity $X(\theta) - q_j$ the function has a kink. Define $O = (0, \bar{X}) \times [0, \bar{X}]$ and $S = [0, \bar{X}] \times [0, \bar{X}]$. Although the function has kinks, it is continuous on any subset of $\mathbb{R}^2$, and hence the profit function is bounded on any compact subset $S$. Therefore the right-hand and left-hand derivatives exist for each state profit function as well as the expected profit function on $O$. The right-hand and left-hand derivatives of the state $\theta$ profit function are certainly finite for $q_i \in (0, X(\theta) - q_j)$ and $q_i > X(\theta) - q_j$, because it is differentiable on these intervals. At $q_i = X(\theta) - q_j$, the right-hand derivative in $q_i$ is zero and the left-hand derivative is finite by Assumption 4. Therefore, the right-hand and left-hand derivatives of $\Pi_i^c$ can be passed inside the expectation (Theorem 16.8, Billingsley 1986).

We first need to show that the right-hand partial derivative in the firm’s own capacity being equal to zero, is a necessary condition for a pure strategy equilibrium. The case where the right-hand derivative is positive is immediately ruled out because there is a profitable upward deviation arbitrarily close to the capacity for both firms. We are left to prove that when the right-hand derivative is negative at $(\bar{q}_1, \bar{q}_2)$, those capacities cannot be an equilibrium. Suppose to the contrary, that at an equilibrium, it is possible that $\partial_x^+ \Pi(\bar{q}_1, \bar{q}_2) < 0$. For notational convenience denote $\bar{q} = \bar{q}_1 + \bar{q}_2$. Define the set of states with kinks at $(\bar{q}_1, \bar{q}_2)$ as $K(\bar{q}_1, \bar{q}_2) = \{ \theta \in \Theta \mid \bar{q} = X(\theta) \}$. For the states $\Theta \setminus K(\bar{q}_1, \bar{q}_2)$, the state revenue is differentiable. For any $\theta \in K(\bar{q}_1, \bar{q}_2)$, the revenue for firm $i$ is zero at $(\bar{q}_1, \bar{q}_2)$ and positive for all $q_i < \bar{q}_i$, and therefore the left-hand derivative of each state revenue is negative at $(\bar{q}_1, \bar{q}_2)$. Putting these two facts together, the left-hand derivative of the expected profit function at $(\bar{q}_1, \bar{q}_2)$ must be less than the right-hand derivative. Hence, there is a profitable deviation $q_i < \bar{q}_i$.

The second part of the proof addresses the necessary symmetry of all equilibria. Suppose to the contrary that there is an equilibrium $(\bar{q}_1, \bar{q}_2)$ that is not symmetric (without loss of generality, assume that $\bar{q}_1 > \bar{q}_2$). Then, for $i \in \{1, 2\}$, $(CE)$ must hold for both firms at $(\bar{q}_1, \bar{q}_2)$. It will be useful to rewrite $(CE)$:

$$\bar{q}_i E [\partial_q P(\bar{q}, \theta) \mid U(\bar{q})] + E [P(\bar{q}, \theta) \mid U(\bar{q})] - a'(\bar{q}_i) = 0,$$

(8)

for $i \in \{1, 2\}$. The condition $(CE)$ holds for both agents, which clearly implies that

$$\partial_x^+ \Pi_a^c(\bar{q}_1, \bar{q}_2) = \partial_x^+ \Pi_a^c(\bar{q}_2, \bar{q}_1).$$

(9)

Since $a$ is convex and $\bar{q}_1 > \bar{q}_2$, $a'(\bar{q}_1) > a'(\bar{q}_2)$. Also, notice that the second term on the right-hand side of (8) is the same for both firms. Based on (8) and (9) we derive the inequalities below,

$$\bar{q}_1 E [\partial_q P(\bar{q}, \theta) \mid U(\bar{q})] \geq \bar{q}_2 E [\partial_q P(\bar{q}, \theta) \mid U(\bar{q})].$$
Since $E[\partial_q P(\tilde{q}, \theta) | U(\tilde{q})] < 0$, the above equality reduces to $\tilde{q}_1 \leq \tilde{q}_2$, a contradiction. ■

**Proof of Lemma 1.** Denote the region $C'(x, y)$, the set of $\theta$ such that $p_1 = p_2 = P(x + y, \theta) > 0$ is a pure strategy equilibrium of the pricing subgame.

Part 1). Suppose that, without loss of generality, $x \geq y$. Now we argue that, for all $r \in \Gamma$, when market clearing prices is an equilibrium, it is the unique equilibrium for any states $\theta$. First we take any other pure pricing strategies $p_1$ and $p_2$.

Case 1). $p_i > P(x + y, \theta)$ and $p_i \geq p_j$. (1a) $p_i > p_j$ : Firm $j$ can increase its profit by pricing at $\min\{(p_i + p_j)/2, P(y, \theta)\}$ and get profit $\min\{(p_i + p_j)/2, P(y, \theta)\}y > p_j \min\{D(p_j), y\}$. (1b) $p_i = p_j$ : Firm $j$ can increase its profit by pricing at $\min\{(p_i - \epsilon), P(y, \theta)\}$ and get profit $\min\{(p_i - \epsilon), P(y, \theta)\}y > p_i D(p_i)/2$, which always holds for for small enough $\epsilon > 0$.

Case 2). $p_j < P(x + y, \theta)$ and $p_i \geq p_j$. Firm $j$ can always increase profit by pricing at $P(x + y, \theta)$ and get $P(x_1 + x_2, \theta)x_j > p_j x_j$.

We have just shown, that the in all demand states $C'(x, y)$, the unique pure strategy equilibrium is symmetric market clearing prices $p_1 = p_2 = P(x + y, \theta) > 0$. It remains to show that in these demand states there are no mixed strategy equilibrium in the pricing subgame.

Based on the preceding analysis, it is straightforward to deduce that the set of firm 1’s best responses to all of firm 2’s prices is $[P(x + y, \theta), P(x, \theta)]$, and similarly for firm 2, the set of all best responses is $[P(x + y, \theta), P(y, \theta)]$. All prices that are part of a mixed strategy Nash equilibrium must be in these sets. We use the fact that mixed strategy Nash equilibrium can only involve best responses to the mixture other firms prices. Take the arbitrary closed sets of prices $S_1 \subseteq [P(x + y, \theta), P(x, \theta)]$ and $S_2 \subseteq [P(x + y, \theta), P(y, \theta)]$. Suppose there is a mixed strategy Nash equilibrium with supports $S_1$ and $S_2$ such that $S_1 \cup S_2 \neq P(x + y, \theta)$. We will show $S_1$ and $S_2$ cannot be the support of a mixed strategy equilibrium by way of contradiction. Recall that, without loss of generality, we imposed that $x \geq y$. Denote $\overline{p}_1 = \max S_1$ and $\overline{p}_2 = \max S_2$. For all $p_2 \in (P(x + y, \theta), \overline{p}_2]$, the best response of firm 1 is to price less than $p_2$. This is because $(p_2 - \epsilon)x > p_2 \max\{D(p_2, \theta)/2, D(p_2, \theta) - y\}$ for small enough $\epsilon > 0$. This must be true because $x > \max\{D(p_2, \theta)/2, D(p_2, \theta) - y\}$. Therefore, $\max S_1 < \max S_2$. But, for all $p_1 \in (P(x + y, \theta), \overline{p}_1]$, the best response of firm 2 is to price weakly less than $p_1$. We will show that for all $p_2 \geq p_1$, $p_2 \min\{D(p_2, \theta) - x, y\} \leq p_1 \min\{D(p_1, \theta) - x, y\}$. The unique maximizer of $p_2 \min\{(D(p_2, \theta) - x), y\}$ is $P(x + y, \theta)$. Based on our assumptions $p_2 \min\{D(p_2, \theta) - x, y\}$ is concave in $p_2$. Putting the above facts together, we know that $p_2 \min\{(D(p_2, \theta) - x), y\}$ is weakly decreasing in $p_2$, for all $p_2 > P(x + y, \theta)$. Therefore, $\max S_1 \geq \max S_2$, which contradicts $\max S_1 < \max S_2$.

Part 2). First we show that $C^p(x, y) \subseteq C'(x, y)$. Fix $(x, y)$ and take any state $\theta \in C^p(x, y)$. This means that symmetric market clearing pricing is the unique equilibrium of the state $\theta$ under the scheme $p$. Any defection to a price higher leads to lower profit than in the case of proportional rationing. Therefore a firm will not find a price increase defection
profit increasing under $r$ unless it also does under $p$. A price decrease gives the firm the same profit. Therefore, under $r$, a firm will not have a profitable defection unless it does under $p$. This implies that any pricing equilibrium with proportional rationing must be pricing equilibrium for all $r \in \Gamma$.

Now we show that $C^r(x, y) \subseteq C^e(x, y)$. Fix $(x, y)$ and take any state $\theta \in C^e(x, y)$. Any defection to a price higher leads to higher profit than in the case of efficient rationing. Therefore a firm will not find a price increase profit increasing under $e$ unless it also does under $r$. A price decrease gives the firm the same profit. Therefore, under $e$, a firm will not have a profitable defection unless it does under $e$. Therefore, $C^r(x, y) \subseteq C^e(x, y)$.

**Proof of Lemma 2.** First take any $(x, y)$ such that $x \geq y$, for all $\theta \in M^r_a(x, y)$, firm 1 can always guarantee itself at least the expected revenue $R(y, \theta) = \max\{p(D(p, \theta) - y)\}$. The expected revenue of a mixed strategy equilibrium of the pricing subgame cannot result in lower expected revenue than $R(y, \theta)$, otherwise a profitable defection exists to the price $p_a = P(r(y, \theta) + y, \theta)$, which results in at least the expected revenue $R(y, \theta)$. Notice that is exactly equal to efficient rationing equilibrium expected revenue, i.e., $R(y, \theta) = v^e(y, \theta)$.

Next take any $(x, y)$ such that $x < y$, for all $\theta \in M^r_a(x, y)$, firm 1 can always guarantee itself at least the expected revenue $p^e(x, y, \theta)x$. This is because any strategy for firm $b$ to price lower than $p^e(x, y, \theta)$ is strictly dominated by $p_b = P(r(x, \theta) + x, \theta)$. No mixed strategy equilibrium can involve strictly dominated strategies. Hence, the expected revenue of a mixed strategy equilibrium of the pricing subgame cannot result in lower expected revenue than $p^e(x, y, \theta)x$, otherwise a profitable defection exists to the price $p^e(x, y, \theta)$.

**Proof of Theorem 2**

**Part 1** The proof part 1 is constructed based on the following two lemmas.

**Lemma 3** Consider the capacity choice game defined by assumptions 1 – 5, with $r = e$. There is a unique symmetric ‘candidate’ subgame perfect equilibrium capacity $\hat{x}$ characterized by the right-hand derivative condition:

$$E[\partial_q P(2\hat{x}, \theta)\hat{x} + P(2\hat{x}, \theta) | C^e(\hat{x}, \hat{x})] - a'(\hat{x}) = 0. \quad (SC)$$

**Proof of Lemma 3.**

An analogous argument to that in Theorem 1 ensures that $\forall \theta \in \Theta$ the right-hand and left-hand derivatives of $\Pi^e_a(x, y)$ in $x$ and $R^e(x, y, \theta)$ in $x$ exist, and are finite on $(0, \hat{X}) \times [0, \hat{X}]$. Hence differentiation can be passed inside the expectation. It is easy to show that the right-hand partial derivative of $\Pi^e_a(x, y)$ in $x$ is the expression on the right-hand side of $(SC)$ by applying Liebniz’s rule and including the fact that the right-hand derivative of the
revenue of any state not in \( C^e(\hat{x}, \tilde{x}) \) is zero. The right-hand partial derivative of \( \Pi_a^e(x, y) \) in \( x \) always exists. For any \( x > y \), the profit is twice-continuously differentiable, so \( \partial_{x}^{2} \Pi_a^e(x, y) = \partial_{x} \Pi_a^e(x, y) \). For all \( x > y \), since the second derivative of the profit function is negative left-hand partial derivative at \( x \) means there must be a profitable local defection from \( \hat{x} \) for both firms.

\[
E \left[ \partial_{q}^{2} P(2\hat{x}, \theta) \hat{x} + 2\partial_{q} P(2\hat{x}, \theta) \right] C^e(\hat{x}, \tilde{x}) \right] = a''(\hat{x}).
\]  

(10)

For all \( \theta \), \( \partial_{q}^{2} P(2\hat{x}, \theta) \hat{x} + 2\partial_{q} P(2\hat{x}, \theta) < 0 \) based on the facts that \( P \) is strictly decreasing and concave. \( E \left[ \partial_{q}^{2} P(2\hat{x}, \theta) \hat{x} + 2\partial_{q} P(2\hat{x}, \theta) \right] C^e(\hat{x}, \tilde{x}) \} < 0 \). Further the cost function is convex, \(-a''(\hat{x}) < 0 \). Therefore the expression (10) is negative, a sufficient condition for the concavity of \( \Pi_a^e(x, y) \) for all \( x > y \).

Based on these two facts, it is necessary that any pure symmetric capacity equilibrium satisfy (11).

\[
E \left[ \partial_{q} P(2\hat{x}, \theta) \hat{x} + P(2\hat{x}, \theta) \right] C^e(\hat{x}, \tilde{x}) \right] = a'(\hat{x}) \leq 0.
\]  

(11)

Next we establish that if (11) holds with strict inequality at \( x = y = \hat{x} \), then \( \hat{x} \) cannot be a capacity choice game equilibrium. We do this by showing that a negative right-hand partial derivative at \( \hat{x} \) implies that that left-hand partial derivative at \( \hat{x} \) is also negative. A negative left-hand partial derivative at \( \hat{x} \) means there must be a profitable local defection from \( \hat{x} \) for both firms.

We break the parameters \( \theta \) into two sets

\[
M^e(\hat{x}, \tilde{x}) \cup \max B(\hat{x}, \tilde{x}) \text{ and } \Theta \setminus (M^e(\hat{x}, \tilde{x}) \cup \max B(\hat{x}, \tilde{x})).
\]

For all \( \theta \in \Theta \setminus (M^e(\hat{x}, \tilde{x}) \cup \max B(\hat{x}, \tilde{x})) \), the state revenue is differentiable at \( (\hat{x}, \tilde{x}) \). For all states \( \theta \in M^e(\hat{x}, \tilde{x}) \cup \max B(\hat{x}, \tilde{x}) \), the right-hand partial derivative of the state revenue function for both firms is zero, while the left-hand partial derivative at \( (\hat{x}, \tilde{x}) \) for both firms is negative. We show that left-hand partial derivative at \( (\hat{x}, \tilde{x}) \) for all \( \theta \in M^e(\hat{x}, \tilde{x}) \cup \max B(\hat{x}, \tilde{x}) \) separately for two cases (i) \( D(p^e(\hat{x}, \tilde{x}, \theta), \theta) \geq \hat{x} \) and (ii) \( D(p^e(\hat{x}, \tilde{x}, \theta), \theta) \leq \hat{x} \).

In case (i), for all \( \theta \in M^e(\hat{x}, \tilde{x}) \cup \max B(\hat{x}, \tilde{x}) \) the left-hand derivative for each at \( (\hat{x}, \tilde{x}) \) is:

\[
\frac{\partial_x r(\hat{x}, \theta) \hat{x}}{\hat{x}} \left( \partial_x P(\hat{x} + r(\hat{x}, \theta), \theta) r(\hat{x}, \theta) + P(\hat{x} + r(\hat{x}, \theta), \theta) \right) \right)
\]

\( * \)

\[+ \frac{r(\hat{x}, \theta)}{\hat{x}} \partial_x P(\hat{x} + r(\hat{x}, \theta), \theta) \hat{x} + P(\hat{x} + r(\hat{x}, \theta), \theta) \right].
\]

The term \( * \) is zero, based on the definition of \( r(\hat{x}, \theta) \) as the zero cost Cournot best response to \( \hat{x} \). The expression becomes:

\[
\frac{r(\hat{x}, \theta) \partial_x P(\hat{x} + r(\hat{x}, \theta), \theta) \hat{x} + P(\hat{x} + r(\hat{x}, \theta), \theta) \right)}{\hat{x}} < 0.
\]  

(12)
The expression in (12) is negative because \( r(\tilde{x}, \theta)/\tilde{x} > 0 \) and \((**)\) is negative; based on the fact that \( \tilde{x} > r(\tilde{x}, \theta) \) and \( P(x, \theta)x \) is strictly concave in \( x \).

Denote \( D = D(p^e(\tilde{x}, \tilde{x}, \theta), \theta) + p^e(\tilde{x}, \tilde{x}, \theta)\partial_p D(p^e(\tilde{x}, \tilde{x}, \theta), \theta) \). Then for case (ii) for all \( \theta \in M^e(\tilde{x}, \tilde{x}) \cup \max B(\tilde{x}, \tilde{x}) \) the left-hand derivative for each at \( (\tilde{x}, \tilde{x}) \) is:

\[
\partial_x r(\tilde{x}, \theta) \left( \frac{\tilde{x}}{D} \right) (\partial_x P(\tilde{x} + r(\tilde{x}, \theta), \theta)r(\tilde{x}, \theta) + P(\tilde{x} + r(\tilde{x}, \theta), \theta) \\
+ \frac{r(\tilde{x}, \theta)}{D} \partial_x P(\tilde{x} + r(\tilde{x}, \theta), \theta) + \frac{r(\tilde{x}, \theta)}{D(p^e(\tilde{x}, \tilde{x}, \theta), \theta)} P(\tilde{x} + r(\tilde{x}, \theta), \theta) \\
< \partial_x r(\tilde{x}, \theta) \left( \frac{\tilde{x}}{D} \right) (**) + \frac{r(\tilde{x}, \theta)}{D(p^e(\tilde{x}, \tilde{x}, \theta), \theta)} (**) \\
< 0.
\]

The first inequality comes from the fact that

\[
D(p^e(\tilde{x}, \tilde{x}, \theta), \theta) + p^e(\tilde{x}, \tilde{x}, \theta)\partial_p D(p^e(\tilde{x}, \tilde{x}, \theta), \theta) < D(p^e(\tilde{x}, \tilde{x}, \theta), \theta),
\]

and the final inequality follows immediately from the fact that \((*) = 0\) and \((** *) < 0\).

We have now shown that the left-hand partial derivative for all states \( \theta \in M^e(\tilde{x}, \tilde{x}) \cup \max B(\tilde{x}, \tilde{x}) \) is negative. Therefore the profit is differentiable at \((\tilde{x}, \tilde{x})\) if \( \mu(M^e(\tilde{x}, \tilde{x}) \cup \max B(\tilde{x}, \tilde{x})) = 0 \), otherwise and is left-hand partial derivative is negative at \((\tilde{x}, \tilde{x})\).

The final step in the proof is to show that a unique \( \tilde{x} \) satisfies \((SC)\). Suppose to the contrary that there is \( q^o \neq \tilde{x} \) that is a pure symmetric equilibrium capacity. If \( q^o > \tilde{x} \), then \( C^e(q^o, q^o) \subset C^e(\tilde{x}, \tilde{x}) \), and by the strict concavity of the the revenue function for any \( \theta \in C^e(q^o, q^o) \) and \( q^o > \tilde{x} \) we have that

\[
\partial_q P(2q^o, \theta)q^o + P(2q^o, \theta) < \partial_q P(\tilde{x} + q^o, \theta)\tilde{x} + P(\tilde{x} + q^o, \theta) \\
< \partial_q P(2\tilde{x}, \theta)\tilde{x} + P(2\tilde{x}, \theta).
\]

The second inequality is based on the fact that the inverse demand function is strictly decreasing and concave in \( x \). Hence, the marginal revenue of \((q^o, q^o)\) on \( C^e(q^o, q^o) \) is greater than \( \tilde{x} \) on \( C^e(q^o, q^o) \) and we have:

\[
E \left[ \partial_q P(2q^o, \theta)q^o + P(2q^o, \theta) \right] C^e(q^o, q^o) < E \left[ \partial_q P(2\tilde{x}, \theta)\tilde{x} + P(2\tilde{x}, \theta) \right] C^e(q^o, q^o) \\
\leq E \left[ \partial_q P(2\tilde{x}, \theta)\tilde{x} + P(2\tilde{x}, \theta) \right] C^e(\tilde{x}, \tilde{x})
\]

The second inequality holds because \( C^e(q^o, q^o) \subset C^e(\tilde{x}, \tilde{x}) \) and that for all \( \theta \in C^e(\tilde{x}, \tilde{x}) \) the state marginal revenue is non-negative. Putting the above inequality together with the fact that \( q(\cdot) \) is convex contradicts the necessary condition for \( q^o \) to be a pure strategy symmetric equilibrium. The argument for \( q^o < \tilde{x} \) follows analogously.

The second lemma used to proof Theorem 2 establishes that only the largest Cournot capacity can be an equilibrium of the capacity choice game.
Lemma 4 \( \hat{x} \neq q^* < \bar{q} \).

**Proof of Lemma 4.** Suppose to the contrary that \( \hat{x} = q^* < \bar{q} \). Since \( \bar{q} \) is an uncertain Cournot equilibrium, the following equality must hold:

\[
E \left[ \partial_q P(2\bar{q}, \theta)q + P(2\bar{q}, \theta) \mid U(2\bar{q}) \right] - a'(\bar{q}) = 0.
\]

If we instead take the expectation over \( C^e(\bar{q}, \bar{q}) \), the expectation will be weakly larger because only all of the positive terms remain in the expectation. Hence,

\[
E \left[ \partial_q P(2\bar{q}, \theta)q + P(2\bar{q}, \theta) \mid C^e(\bar{q}, \bar{q}) \right] - a'(\bar{q}) \geq 0. \tag{13}
\]

Since the \( P(q, \theta)q \) is strictly concave,

\[
\partial_q P(q + q^*, \theta)q + P(q + q^*, \theta) > \partial_q P(2\bar{q}, \theta)q + P(2\bar{q}, \theta) \tag{14}
\]

for all \( \theta \in C^e(\bar{q}, \bar{q}) \). Since (14) is true for each \( \theta \in C^e(\bar{q}, \bar{q}) \), from (13) we know that

\[
E \left[ \partial_q P(q + q^*, \theta)q + P(q + q^*, \theta) \mid C^e(\bar{q}, \bar{q}) \right] - a'(\bar{q}) > 0. \tag{15}
\]

By definition of \( C^e, C^e(\bar{q}, \bar{q}) \subset C^e(\bar{q}, q^*) \), and since the marginal revenue for each \( \theta \in C^e(\bar{q}, q^*) \) must be positive we know that

\[
E \left[ \partial_q P(q + q^*, \theta)q + P(q + q^*, \theta) \mid C^e(\bar{q}, \bar{q}) \right] - a'(\bar{q}) \geq 0. \tag{16}
\]

Putting together (15) and (16) we have

\[
E \left[ \partial_q P(q + q^*, \theta)q + P(q + q^*, \theta) \mid C^e(\bar{q}, q^*) \right] - a'(\bar{q}) > 0. \tag{17}
\]

Expression (17) shows the right hand derivative at \( x = \bar{q} > y = q^* \) is positive. \( \Pi^e(x, y) \) is strictly concave for all \( x \geq y \), hence the right-hand derivative at \( x = y = q^* \) must be greater than at \( x = \bar{q} > y = q^* \). This contradicts to the necessary condition \((SC)\) for \( x = y = q^* \). \( \blacksquare \)

Next we prove statement 1 of Theorem 2. We do this separately for efficient and proportional rationing.

*(Efficient Rationing)* Based on Lemma 4, only the UCE \( x = y = \bar{q} \) can be an equilibrium of the efficient rule capacity choice game. Further, from Lemma 3 the only possible symmetric equilibrium is \( x = y = \hat{x} \).

Now we show that only if \( \hat{x} = \bar{q} \), can the symmetric candidate be an equilibrium. Suppose to the contrary that \( x = y = \hat{x} \neq \bar{q} \) is a capacity choice equilibrium. Based on conditions \((CE)\) and \((SC)\), it is only possible that \( \hat{x} > \bar{q} \). This implies that at \( x = y = \hat{x} \) each firm can increase their uncertain Cournot profit by decreasing capacity. Hence, there exists \( z \) such that \( \Pi^e_\theta(z, \hat{x}) > \Pi^e_\theta(\hat{x}, \hat{x}) \). At \( x = y = \hat{x} \) all pricing is Cournot or Bertrand which implies
that $\Pi^c_a(\tilde{x}, \tilde{x}) = \Pi^c_a(\tilde{x}, \tilde{x})$. But firms always earn weakly more profit at any fixed capacities in the capacity choice game than the uncertain Cournot game, $\Pi^c_a(z, \tilde{x}) \geq \Pi^c_a(z, \tilde{x})$. Putting the above fact together, $\Pi^c_a(z, \tilde{x}) \geq \Pi^c_a(\tilde{x}, \tilde{x})$, a contradiction.

**Proportional Rationing** The proof is done by showing that if any UCE other than $\tilde{q}$ is an equilibrium of the capacity choice game it will not have Cournot pricing. Formally, for $r = p$, if $x = y = q^* < \tilde{q}$ is an equilibrium of the capacity choice game, then $\mu(M^p(q^*, q^*)) > 0$.

Suppose to the contrary that $x = y = q^*$ is an equilibrium of the capacity choice game and $\mu(M^e(q^*, q^*)) = 0$. Then, at $(q^*, q^*)$ it must be that $\mu(M^e(q^*, q^*)) = 0$ since, $M^e(q^*, q^*) \subset M^p(q^*, q^*)$. As a result, the three profits are equal,

$$
\Pi^e_a(q^*, q^*) = \Pi^p_a(q^*, q^*) = \Pi^c_a(q^*, q^*).
$$

(18)

Since $(q^*, q^*)$ is a Nash equilibrium of the proportional rationing capacity choice game, it must be that, for all $x \in [0, \bar{X}]$,

$$
\Pi^p_a(x, q^*) \leq \Pi^p_a(q^*, q^*).
$$

(19)

For any capacities $(x, y)$, proportional rationing the equilibrium expected revenue is always weakly greater than with efficient rationing. Hence, for all $x \in [0, \bar{X}]$,

$$
\Pi^c_a(x, q^*) \leq \Pi^p_a(x, q^*).
$$

(20)

Putting (18), (19), and (20) together we have, for all $x \in [0, \bar{X}]$,

$$
\Pi^c_a(x, q^*) \leq \Pi^c_a(q^*, q^*).
$$

Therefore, $x = y = q^*$ must be a Nash equilibrium of the capacity choice game with efficient rationing. But we know from Part 2 this cannot be true. Hence, we have a contradiction.

**Part 2** For the proof of part 2, one general proof covers for both rationing schemes.

**Necessity** For each $r \in \{e, p\}$, if $(\bar{q}, \bar{q})$ is an equilibrium with Cournot pricing, then, trivially, $\bar{q} \in \beta^r_a(\bar{q})$ and $\mu(M^r(\bar{q}, \bar{q})) = 0$.

**Sufficiency** For each $r \in \{e, p\}$, we claim that, if $\exists x \in \beta^r_a(\bar{q})$ such that $\mu(M^r(x, \bar{q})) = 0$, then $(\bar{q}, \bar{q})$ is an equilibrium. Suppose on the contrary, that $\exists x \in \beta^r_a(\bar{q})$ such that $\mu(M^r(x, \bar{q})) = 0$ and $\bar{q} \notin \beta^r_a(\bar{q})$. By definition $\bar{q} \in \beta^r_a(\bar{q})$, and therefore, for all $x \in \beta^r_a(\bar{q})$, $\Pi^c_a(\bar{q}, \bar{q}) \geq \Pi^c_a(x, \bar{q})$. Since, $\mu(M^r(x, \bar{q})) = 0$, $\Pi^c_a(x, \bar{q}) = \Pi^c_a(x, \bar{q})$. At any fixed capacities, the expected revenue in the capacity choice game is weakly higher than in the uncertain Cournot game. Thus, $\Pi^c_a(\bar{q}, \bar{q}) \geq \Pi^c_a(x, \bar{q})$. Putting the facts above together, $\Pi^c_a(\bar{q}, \bar{q}) \geq \Pi^c_a(x, \bar{q})$ for all $x \in \beta^r_a(\bar{q})$ such that $\mu(M^r(x, \bar{q})) = 0$. Therefore, $\bar{q} \in \beta^r_a(\bar{q})$, a contradiction. ■
Proof of Theorem 3  The first step of the proof is a generalization of the 1st Judo Principle from Lepore (2009).

Lemma 5 (Extended 1st Judo Principle ) Assume \( r = e \). If \( \mu(C^e(q,\bar{q})) = 1 \), then, for all \( y \in [0,\bar{X}] \), firm \( a \) will never find it optimal to choose \( x \) such that there is positive probability on parameters in \( M^e_a(x,y) \); that is,

\[
\forall x \in \beta^e_a(y) \implies \mu(M^e_a(x,y)) = 0.
\]  

(21)

Proof of Lemma 5. We base our proof on the fact that if \( x \geq y \) and \( x \in \beta^e_a(y) \), then the right-hand derivative must be zero,

\[
E[\partial_x R^e(x,y,\theta) | C^e(x,y)] + E[\partial_x C^e(x,y) | M^e_a(x,y)] - a'(x) = 0.
\]  

(22)

From Lemma 3 we know that at any kink, \( x = y \), such that \( x \in \beta^e_a(y) \), (22) must be true. At asymmetric capacities \( x > y \) the function is differentiable. Thus, if \( x \in \beta^e_a(y) \), then (22) must hold.

Case 1: Suppose that \( x \geq y \) and \( x + y \leq 2\bar{q} \).

In the case of efficient rationing, the only states with non-zero right-hand derivatives have Cournot pricing. Hence, in both cases (22) can be reduced to,

\[
E[\partial_y P(x+y,\theta)x + P(x+y,\theta) | C^e(x,y)] - a'(x) = 0.
\]  

(23)

Based on Lemma 3-4 and that \( \mu(C^e(q,\bar{q})) = 1 \), (23) can only hold true if \( x = y = \bar{q} \).

Case 2: Suppose that \( x \geq y \) and \( x + y > 2\bar{q} \). We show that if \( x + y > 2\bar{q} \), then \( \partial_x \Pi^e_a(x,y) < 0 \), which implies that \( x \) is not a best response to \( y \).

First we examine the case that \( x \geq y > \bar{q} \). This means that for all \( \theta \in C^e(\bar{q},\bar{q}) \) that \( \theta \in C^e(x,y) \cup M^e(x,y) \cup B(x,y) \). If \( \theta \in C^e(x,y) \), then based on the concavity of \( P \), \( \partial^e_y P(x+y,\theta)x < \partial^e_y P(2\bar{q},\theta)\bar{q} \) and \( P(x+y,\theta) \geq P(2\bar{q},\theta) \), which implies the marginal revenue of that state decrease. If \( \theta \in B(x,y) \), then the marginal revenue goes to zero, a decrease from positive marginal revenue at \((\bar{q},\bar{q})\). With regards to \( \theta \in M^e(x,y) \), we must analyze the two rationing schemes separately. For efficient rationing, the marginal revenue goes to zero, a decrease from positive marginal revenue at \((\bar{q},\bar{q})\). Putting together that the marginal cost is weakly decreasing in \( x \) with the fact that for all \( \theta \in C^e(\bar{q},\bar{q}) \), \( \partial_x R^e(\bar{q},\bar{q},\theta) > \partial_x R^e(x,y,\theta) \), it follows that \( \partial_x \Pi^e_a(x,y) < \partial_x \Pi^e_a(\bar{q},\bar{q}) < 0 \).

In order to complete the proof of Theorem 3, we show if the 1st Judo Principle is true, then unique equilibrium is the UCE capacities.

First notice that Condition (21) implies Condition (7). This is because Condition (21) implies that each firm’s best response to \( \bar{q} \) must be in the Cournot or Bertrand. If \( \bar{q} \in \beta^e_a(\bar{q}) \),
then \( q \in \beta_a(\bar{q}) \) and because \( x = y = \bar{q} \) is symmetric, \( \mu (\mathcal{M}^e(\bar{q}, \bar{q})) = 0 \). Hence, \( x = y = \bar{q} \) is an equilibrium.

To show uniqueness, notice there cannot be an equilibrium with mixed pricing because one firm would have to be weakly larger, and this would imply \( \mu (\mathcal{M}^e(x, y, \bar{q})) = 1 \) cannot be an equilibrium because \( x^e = 0 \) would be a profitable deviation. Therefore, there can only be an equilibrium \( (x, y) \) such that \( \mu (\mathcal{C}^e(x, y, \bar{q})) > 0 \) and \( \mu (\mathcal{M}^e(x, y, \bar{q})) = 0 \). This implies that any equilibrium must satisfy \((CE)\). Hence, it must be symmetric and we know from Theorem 2 that the only possible equilibria with UCE capacities is \( x = y = \bar{q} \).

**Proof of Proposition 1.** Define the demand parameter

\[
T(q_1 + q_2) = \max \{ \theta \in [\underline{\theta}, \bar{\theta}] \mid P(q_1 + q_2, \theta) \geq 0 \}.
\]

By Leibniz’s rule the second partial derivative of the expected profit function in \( q \) is

\[
\left| \partial_{q_i}^2 E [P(q_1 + q_2, \theta)q_i] \right| = \int_{T(q_1 + q_2)} \left( \partial_{q_i}^2 P(q_1 + q_2, \theta)q_i + 2\partial_{q_i} P(q_1 + q_2, \theta) \right) d\mu. \tag{24}
\]

Each inverse demand function in the parameter range \( (T(q_1 + q_2), \bar{\theta}) \) is twice-continuously differentiable \( \forall q_i \in (0, X(\bar{\theta}) - q_3) \). The strict concavity of each individual revenue function implies that each second derivative is strictly less than zero and since \( \mu \) is a probability distribution, the integral \( (24) \) must be negative.

Next we show that

\[
\left| \partial_{q_i q_j}^2 E [P(q_1 + q_2, \theta)q_i] \right| < \left| \partial_{q_i}^2 E [P(q_1 + q_2, \theta)q_i] \right|.
\]

We calculate the term on the left-hand side,

\[
\partial_{q_i q_j}^2 E [P(q_1 + q_2, \theta)q_i] = \int_{T(q_1 + q_2)} \left( \partial_{q_i q_j}^2 P(q_1 + q_2, \theta)q_i \right) d\mu,
\]

\[
= \int_{T(q_1 + q_2)} \left( \partial_{q_i}^2 P(q_1 + q_2, \theta)q_i \right) d\mu. \tag{25}
\]

When we compare the expression in \( (25) \) to the expression in \( (24) \), we see that \( (25) \) has one less negative term in the integral. Therefore, the expression in \( (24) \) is more negative than the expression in \( (25) \), which implies

\[
\left| \partial_{q_i q_j}^2 E [P(q_1 + q_2, \theta)q_i] \right| < \left| \partial_{q_i}^2 E [P(q_1 + q_2, \theta)q_i] \right|.
\]
The strict concavity of the payoff functions and the fact that the slope of the best response function is in the interval \((-1, 0]\) are sufficient conditions for the uniqueness of equilibrium.\(^\text{12}\)

References


\(^{12}\)These sufficient conditions for uniqueness of equilibrium are shown in Vives (1999).


