Question 1 This is exactly problem 11 from section 2.2 in the book. Prove that a straight line is the shortest curve that joins two points in $\mathbb{R}^3$. Do this the following way: Let $c : [a, b] \to \mathbb{R}^3$ be an arbitrary curve from $p = c(a)$ to $q = c(b)$. Let $u = (q - p)/\|q - p\|$.

a) Show that if $\sigma$ is a straight line segment from $p$ to $q$, say $\sigma(t) = (1 - t)p + tq$, $0 \leq t \leq 1$, then $L(\sigma) = d(p, q)$.

b) Cauchy-Schwartz implies that $\|c'\| \geq c' \cdot u$. Use this to deduce that $L(c) \geq d(p, q)$.

c) Show that if $L(c) = d(p, q)$, then $c$ is a straight line segment.

Question 2 Now we are going to investigate the same problem using the calculus of variations. Very often in math or physics, one is interested in minimizing or maximizing a functional. For our purposes a functional $F$ will be a function from some set of functions to $\mathbb{R}$. These are often given by integrals. For example, consider the set $C$ of all smooth curves $c$ in the plane joining $p$ to $q$ and parametrized on the interval $[a, b]$. Then the length functional $L$ is $L : C \to \mathbb{R}$ given by

$$L(c) = \int_a^b \|c'\| \, dt$$

If we further assume that $c$ is the graph of a function $y = c(t)$ joining the points $p = (a, c(a))$ to $q = (b, c(b))$, then $L$ can be written as

$$L(c) = \int_a^b \sqrt{1 + (c')^2} \, dt$$

To find the shortest curve joining $p$ to $q$, we would like to “differentiate $L$ with respect to $c$” and set the result equal to 0 to find the “critical curves” which we hope are minimums or shortest curves (geodesics).

Here is the general framework in which to do this. Consider a suitably differentiable function $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by $F(t, x, y)$. We wish to find the maxima/minima of the functional

$$J(c) = \int_a^b F(t, c(t), c'(t)) \, dt$$

(To get the length functional, let $F = \sqrt{1 + y^2}$.)

Now we consider a variation of $c$ with endpoints fixed, that is, a function $\alpha : (-\varepsilon, \varepsilon) \times [a, b] \to \mathbb{R}$ such that $\alpha(0, t) = c(t)$ and $\alpha(u, a) = p$ and $\alpha(u, b) = q$ for all $u \in (-\varepsilon, \varepsilon)$. Note that for fixed $u = u_0$, $\alpha(u_0, t)$ is just a curve joining $p$ to $q$. See the picture. As $u$ varies we get a family of curves which “pass through” $c$ when $u = 0$. Denote the $u$–th curve by $c(u)$. 

![Diagram of curves](image-url)
a) Now it’s your turn to do some stuff. For a variation $\alpha$, show that
\[
\frac{d}{du} \left( J(\alpha(u)) \right) \bigg|_{u=0} = \frac{d}{du} \left| \int_{a}^{b} F(t, \alpha(u), \frac{\partial \alpha}{\partial t}(u, t)) \, dt \right|
\]
\[
= \int_{a}^{b} \left[ \frac{\partial \alpha}{\partial u}(0, t) \frac{\partial F}{\partial x}(t, c(t), c'(t)) + \frac{\partial^{2} \alpha}{\partial u \partial t}(0, t) \frac{\partial F}{\partial y}(t, c(t), c'(t)) \right] \, dt
\]
Since mixed partials are equal, $\frac{\partial^{2} \alpha}{\partial u \partial t} = \frac{\partial^{2} \alpha}{\partial t \partial u}$, apply integration by parts to the second term in the integrand and use the fact that endpoints are fixed to conclude
\[
\frac{d}{du} \left( J(\alpha(u)) \right) \bigg|_{u=0} = \int_{a}^{b} \left[ \frac{\partial \alpha}{\partial u}(0, t) \frac{\partial F}{\partial x}(t, c(t), c'(t)) - \frac{d}{dt} \left( \frac{\partial F}{\partial y}(t, c(t), c'(t)) \right) \right] \, dt
\]
b) Thus critical points of $J$ correspond to curves $c$ with
\[
\frac{\partial F}{\partial x}(t, c(t), c'(t)) - \frac{d}{dt} \left( \frac{\partial F}{\partial y}(t, c(t), c'(t)) \right) = 0
\]
This is called the Euler-Lagrange equation of the functional $J$. Use this to show that straight lines are critical points of the length functional $L$. ($F(t, x, y) = \sqrt{1 + y'^2}$.) To show these are actually minima we would have to compute the second derivative of $J$ with respect to $u$ and use the second derivative test. This can be done, but is a big mess!
c) Suppose now that you wanted to find a curve $c$ given as a graph $y = c(t)$ over $[a, b]$, for which the surface of revolution obtained by rotating $c$ about the $t$–axis has minimal area amongst all curves joining $(a, c(a))$ to $(b, c(b))$. To make the problem interesting we assume that $c(t) > 0$ on $[a, b]$. This will give a so-called minimal surface of revolution. What should the function $F$ be, so that the corresponding functional $J$ represents the area of the surface of revolution? Deduce that a curve $c$ that generates a minimal surface of revolution satisfies the non-linear differential equation
\[
1 + \left( \frac{dc}{dt} \right)^2 - c(t) \left( \frac{d^2 c}{dt^2} \right) = 0
\]
Miraculously, this differential equation can be solved since the independent variable $t$ is missing using some standard tricks. See, for example, the Boyce–DiPrima book on differential equations. It turns out that the solution to this differential equation is $c(t) = C \cosh \left( \frac{t + K}{C} \right)$, where $C$ and $K$ are constants. The resulting surfaces are called catenoids.