Econ 522 Midterm (Winter 2016)

1. Consider the time series

\[ y_t = \delta + y_{t-1} + w_t, \]

with initial condition \( y_0 = 0 \) and \( w_t \) is white noise with variance \( \sigma^2 \). Is \( \{ y_t \} \) stationary? What is the mean function, \( \mu_y(t) \)? What is the autocovariance function, \( \gamma_y(s,t) \)?

Solution:
Write

\[ y_t = t\delta + \sum_{i=1}^{t} w_i. \]

Then,

\[ \mu_y(t) = E(y_t) = t\delta \]

and

\[ \gamma_y(s,t) = \text{cov}(y_s, y_t) = E \left( \sum_{i=1}^{s} w_i \right) \left( \sum_{j=1}^{t} w_j \right) = \min\{s,t\} \sigma^2. \]

Not stationary since \( \mu_y \) is a function of \( t \).
2. Consider the time series

\[ x_t = \phi x_{t-1} + w_t \]
\[ y_t = \alpha + w_t + u_t + \theta u_{t-1} \]

where \(|\phi| < 1\) and \(\{w_t\}\) and \(\{u_t\}\) are independent white noise processes with variance \(\sigma_w^2\) and \(\sigma_u^2\) respectively.

(a) What is the autocovariance of \(\{x_t\}\), \(\gamma_x(h)\)?
(b) What is the autocovariance of \(\{y_t\}\), \(\gamma_y(h)\)?
(c) What is the crosscorrelation of \(\{x_t\}\) and \(\{y_t\}\), \(\rho_{xy}(h)\)?

Solution:

(a) \(\gamma_x = \phi^h \sigma_w^2 / (1 - \phi^2)\).

(b) \(\gamma_y = \begin{cases} \sigma_w^2 + (1 + \theta^2) \sigma_u^2 & h = 0 \\ \theta \sigma_u^2 & h = \pm 1 \\ 0 & \text{else} \end{cases}\)

(c) \(\gamma_{xy}(h) = E[x_{t+h}y_t] = \text{cov} \left( \sum_{i=0}^{\infty} \phi^i w_{t+h-i}, \alpha + w_t + u_t + \theta u_{t-1} \right) = \phi^h \sigma_w^2.\)
3. Consider the model

\[ y_t = \alpha + 0.6y_{t-1} + 0.7y_{t-2} + w_t + 2w_{t-1}, \]

where \( w_t \) is white noise with variance \( \sigma_w^2 \). Is this model causal? Is it invertible? (Why?)

**Solution:**

The AR polynomial, \( \phi(z) = 1 - 0.6z - 0.7z^2 \), has roots \( z = (0.841, -1.698) \). One is inside and one is outside the unit circle, so the model is noncasual.

The MA polynomial, \( \theta(z) = 1 + 2z \) has root \( z = -0.5 \), which is inside the unit circle, so noninvertible.
4. [3_01.tex] Consider the model

\[ y_t = \phi y_{t-1} + w_t + \theta w_{t-1}, \]

where \( w_t \) is white noise with variance \( \sigma^2 \).

(a) What conditions on \( \phi \) and \( \theta \) imply that this model is stationary and causal?

(b) Write this model in form \( y_t = \sum_{i=0}^{\infty} \psi_i w_{t-i} \). (Show how to compute \( \psi_i \) \( (i = 1, 2, \ldots) \) in terms of \( \phi \) and \( \theta \).)

Solution:

(a) Causal iff \( |\phi| < 1 \). Invertible iff \( |\theta| < 1 \).

(b) Given

\[ \phi(B)y_t = \theta(B)w_t \]  

(1)

we search \( \psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots \) such that

\[ y_t = \psi(B)w_t. \]  

(2)

Substituting (2) into (1), we get

\[ \phi(B)\psi(B)w_t = \theta(B)w_t, \]

which implies

\[ \phi(z)\psi(z) = \theta(z). \]

Writing this out explicitly, we get

\[ (1 - \phi z)(1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \ldots) = 1 + \theta z. \]

Matching coefficients we get,

\[ \psi_1 - \phi = \theta \]
\[ \psi_2 - \phi\psi_1 = 0 \]
\[ \psi_3 - \phi\psi_2 = 0 \]
\[ \psi_4 - \phi\psi_3 = 0 \]
\[ \vdots \]
\[ \psi_n - \phi\psi_{n-1} = 0 \]
\[ \vdots \]

Thus, the coefficients satisfy the difference equation \( \psi_n - \phi\psi_{n-1} = 0 \) with initial condition \( \psi_1 = \theta + \phi \).