1) In this problem we have $K \subseteq U$, where $K$ is compact and $U$ is open. We are to prove that there exists $\varepsilon > 0$ such that for every $k \in K$ we have $(k - \varepsilon, k + \varepsilon) \subseteq U$. We proceed by contradiction.

Suppose no such $\varepsilon$ exists. Then, in particular, there is some $k_1 \in K$ such that $(k_1 - 1, k_1 + 1) \not\subseteq U$. Similarly, there is some $k_2 \in K$ such that $(k_2 - 1/2, k_2 + 1/2) \not\subseteq U$. In this fashion, we may choose for any $n$ an element $k_n \in K$ such that $(k_n - 1/n, k_n + 1/n) \not\subseteq U$. This sequence $\{k_n\}$ is contained in $K$. Since $K$ is compact, there exists a subsequence $\{k_{n_j}\}$ that converges to some $\beta \in K$. But for each $n$ we have $(k_n - 1/n, k_n + 1/n) \not\subseteq U$, so for each $n$ there exists a point $x_n \in (k_n - 1/n, k_n + 1/n) \cap \mathbb{R} \setminus U$. Since $|x_n - k_n| < 1/n$ we must have $x_{n_j} \rightarrow \beta$ and since $\{x_n\}$ is contained in the closed set $\mathbb{R} \setminus U$, we must have $\beta \in \mathbb{R} \setminus U$. This is a contradiction since $\beta \in K \subseteq U$.

2) (a) Here we have $K$ compact and $T$ closed, with $K \cap T = \emptyset$. So $K$ is contained in the open set $\mathbb{R} \setminus T$. Now apply the result of problem one: there exists $\varepsilon > 0$ such that for every $k \in K$, $(k - \varepsilon, k + \varepsilon) \subseteq \mathbb{R} \setminus T$. In other words, no $k \in K$ is within $\varepsilon$ of any $y \in T$, so $d(K, T) \geq \varepsilon > 0$. □

(b) As for the counterexample when $K$ is assumed merely to be closed, try

$$K = \mathbb{N} \text{ and } T = \{n + 1/(n + 1) : n \in \mathbb{N}\}$$

Then $K$ and $T$ are closed, $K \cap T = \emptyset$, and there is NO positive number $\delta$ such that $|s - t| > \delta$ for all $s \in S$ and $t \in T$. So $d(K, T) = 0$. □

3) By the Heine-Borel Theorem, $K \subseteq U_1 \cup U_2 \cup \ldots \cup U_N$ for some finitely many of the original $U_n$ (the third edition version of the problem just starts with these finitely many $U_j$ to begin with). Now, $U_j$ is open for each $j = 1, 2, \ldots, N$, so we may write

$$U_j = \bigcup_{k=1}^{\infty} (a_{jk}, b_{jk})$$

since every open set is a countable union of disjoint open intervals. Now, the family $\{(a_{jk}, b_{jk}) : j = 1, 2, \ldots, N, k \in \mathbb{N}\}$ is again an open cover of $K$, so again the Heine-Borel Theorem implies that $K$ is actually contained in some finitely many of the intervals $\{(a_{jk}, b_{jk})\}$, say (after relabeling)

$$K \subseteq (a_1, b_1) \cup (a_2, b_2) \cup \ldots \cup (a_M, b_M).$$
We next define yet another open cover. Let $O_n$ be the open set defined by

$$O_n = (a_1 + 1/n, b_1 - 1/n) \cup (a_2 + 1/n, b_2 - 1/n) \cup \ldots \cup (a_M + 1/n, b_M - 1/n)$$

Here we must be careful to only use $n$ big enough so that this definition makes sense. For example, if $(a_1, b_1)$ happens to be $(0, 1/5)$, then $(a_1 + 1/n, b_1 - 1/n)$ makes sense only if $n \geq 6$. Thus we check the lengths of the finitely many intervals $(a_1, b_1), (a_2, b_2), \ldots, (a_M, b_M)$ and pick $n$ big enough so that $O_n$ makes sense.

Now,

$$\bigcup_n O_n = (a_1, b_1) \cup (a_2, b_2) \cup \ldots \cup (a_M, b_M)$$

so $\{O_n\}$ is an open cover for $K$. One last application of Heine-Borel yields a finite subcover $\{O_{n_1}, O_{n_2}, \ldots, O_{n_L}\}$. Since the sets $O_n$ are “nested” (i.e. $O_n \subset O_{n+1}$), we find that in fact $K \subset O_{n_L}$.

Thus, set $\varepsilon = 1/(n_L + 1)$. Then for every $k \in K$,

$$(k - \varepsilon, k + \varepsilon) \subset (a_j, b_j)$$

for some $j = 1, 2, \ldots, M$ (Draw a picture!!). But each interval $(a_j, b_j)$ is itself entirely contained in some one of the original sets $U_\alpha$!!  □

Note: Many of you gave shorter, more direct proofs for this exercise. This is perfectly fine. I’m somewhat attached to this “triple Heine-Borel” argument, though, so I thought you could use it to compare and contrast different approaches. Particularly the last application of H-B where we “exhaust” intervals from within by nested smaller intervals is worth noting.

4) Here we have $K$ compact and $T$ closed. Since $K$ is closed and $T$ is closed, $K \cap T$ is closed too. Since $K$ is bounded, $K \cap T$ is bounded too. Thus, $K \cap T$ is closed and bounded, hence compact. □

5) This one is similar to the previous problem; any intersection of closed sets is closed and any intersection of bounded sets is bounded. Thus any intersection of compact sets is compact. □

6) We are asked to show that any open interval can be written as a union of compact sets. Let $I$ be an open interval.
Case 1: $I = (a, b)$

In this case, for each $n \in \mathbb{N}$ let $K_n = [a + 1/n, b - 1/n]$. Clearly each $K_n$ is compact and

$$\bigcup_{n \in \mathbb{N}} K_n = I.$$

Case 2: $I = (a, \infty)$

In this case, for each $n \in \mathbb{N}$ let $K_n = [a + 1/n, a + n]$. Clearly each $K_n$ is compact and

$$\bigcup_{n \in \mathbb{N}} K_n = I.$$

Case 3: $I = (-\infty, b)$

In this case, for each $n \in \mathbb{N}$ let $K_n = [b - n, b - 1/n]$. Clearly each $K_n$ is compact and

$$\bigcup_{n \in \mathbb{N}} K_n = I.$$

Case 4: $I = (-\infty, \infty)$

In this case, for each $n \in \mathbb{N}$ let $K_n = [-n, n]$. Clearly each $K_n$ is compact and

$$\bigcup_{n \in \mathbb{N}} K_n = I. \quad \square$$