Part I.

1. Circle T for true and F for false.

   a) When Lagrange multipliers are used to obtain the maximum value of the function
      \( f(x, y, z) = yz + xy \) subject to the constraints \( xy = 1 \) and \( y^2 + z^2 = 1 \), one of the
equations that must be solved is \( x + z = \lambda x + 2 \mu z \).

   b) If \( R = [0, 1] \times [0, 1] \), then \( 0 \leq \iint_R \sin(x + y) \, dA \leq 1 \).

   c) If \( f(x, y) = y/(x + 2) \), then \( \int_0^4 f(x, y) \, dy = 8/(x + 2) \).

   d) If \( D \) is the planar region enclosed by \( x = 0 \) and \( x = \sqrt{1 - y^2} \), then
      \[ \iint_D xy^2 \, dA = \int_0^1 \int_0^{\sqrt{1-y^2}} xy^2 \, dy \, dx. \]

   e) If \( D \) is the planar region described in part d, then
      \[ \iint_D xy^2 \, dA = \int_{-\pi/2}^{\pi/2} \int_0^1 r^4 \cos(\theta) \sin^2(\theta) \, dr \, d\theta. \]

   f) The vector field \( F(x, y) = (x - 2x \sin(y)) \mathbf{i} + x^2 \cos(y) \mathbf{j} \) is conservative.

2. a) Write an integral formula in Cartesian coordinates for the \( x \)-coordinate, \( \bar{x} \), of the center of mass of the semi-
circular planar lamina of constant density that is enclosed
by \( x = 0 \) and \( x = \sqrt{1 - y^2} \).

   \[ \bar{x} = \frac{2}{\pi} \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x \, dx \, dy \]

   b) Write an integral formula in cylindrical coordinates for the \( z \)-coordinate, \( \bar{z} \), of the center of mass of the solid
of constant density that is bounded below by the cone
\( z = \sqrt{x^2 + y^2} \) and above by the plane \( z = 2 \). Hint. The
volume of a cone is \( \frac{1}{3} \pi R^2 H \).

   \[ \bar{z} = \frac{3}{8\pi} \int_0^{2\pi} \int_0^2 \int_0^z r \, dr \, dz \, d\theta \]

   c) Write an integral formula, in Cartesian coordinates, for the
area of the part of the surface \( z = xy \) that lies within
the cylinder \( x^2 + y^2 = 1 \).

   \[ A(S) = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1+y^2+x^2} \, dy \, dx \]

   d) Write an integral formula, in polar coordinates, for the
area of the part of the surface \( z = xy \) that lies within the
cylinder \( x^2 + y^2 = 1 \).

   \[ A(S) = \int_0^{2\pi} \int_0^1 \sqrt{1+r^2} \, r \, dr \, d\theta \]

   e) Evaluate the surface area integral in part d of this problem (no calculator). If you need more room, then
use the back of this sheet.

   Use the substitution \( u = 1 + r^2 \) in the \( dr \) integral.

   \[ \int_0^{2\pi} \int_0^1 \sqrt{1+r^2} \, r \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{1}{2} (1+r^2)^{3/2} \right]_0^1 \, d\theta \]

   \[ = \frac{1}{3} \int_0^{2\pi} 2^{3/2} - 1 \, d\theta \]

   \[ = \frac{2\pi}{3} (2\sqrt{2} - 1) \]
Part II.

1. Do all three parts. No calculator, no computer.

   a) Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = x^2 y$ on the ellipse $x^2 + 2y^2 = 6$.

   The max/min values of $f$ are attained when $\nabla f(x, y) = \lambda \nabla g(x, y)$ or $\langle 2xy, x^2 \rangle = \lambda \langle 2x, 4y \rangle$. Therefore, $x$, $y$, and $\lambda$ must satisfy the following three equations.

   $2xy = 2\lambda x$
   $x^2 = 4\lambda y$
   $x^2 + 2y^2 = 6$

   If $\lambda = 0$, then $x = 0$ so $2y^2 = 6$. Consequently, $y = \pm \sqrt{3}$. Observe that $f(0, \pm \sqrt{3}) = 0$.

   If $\lambda \neq 0$ and $x \neq 0$, then $y \neq 0$. Divide the first equation by the second to find that $2y/x = x/2y$ so $x^2 = 4y^2$. It follows that $6y^2 = 6$ so $y = \pm 1$ and $x = \pm 2$. Consequently, $f(\pm 2, 1) = 4$ is the maximum value and $f(\pm 2, -1) = -4$ is the minimum value.

   b) Evaluate the integral $\int\int_D \frac{y}{1 + x^2} \, dA$ where $D$ is bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$.

   The region $D$ is shown on the right. Observe that

   $0 \leq x \leq 1$
   $0 \leq y \leq \sqrt{x}$.

   Therefore, the integral iterates, and evaluates, as follows.

   $$\int_0^1 \int_0^{\sqrt{x}} \frac{y}{1 + x^2} \, dy \, dx = \int_0^1 \left[ \frac{y^2/2}{1 + x^2} \right]_{y=0}^{y=\sqrt{x}} \, dx = \frac{1}{2} \int_0^1 \frac{x}{1 + x^2} \, dx = \frac{1}{4} \ln(1 + x^2)|_0^1 = \ln(2)/4$$

   c) Evaluate the line integral $\int_C x \, dx + (x - y^2) \, dy$ where $C$ is the graph of the equation $x = \sqrt{y}$ from the point $(0, 0)$ to the point $(1, 1)$.

   The path can be parametrized using $x = \sqrt{y}$, $y = y$, $0 \leq y \leq 1$. Since $dx = 1/(2\sqrt{y}) \, dy$, this yields

   $$\int_C x \, dx + (x - y^2) \, dy = \int_0^1 \sqrt{y} \cdot \frac{1}{2\sqrt{y}} + \sqrt{y} - y^2 \, dy$$
   $$= \int_0^1 1/2 + \sqrt{y} - y^2 \, dy$$
   $$= \left[ \frac{1}{2}y + \frac{2}{3}y^{3/2} - \frac{1}{3}y^3 \right]_0^1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$
2. Do both parts.

a) Calculate the iterated integral \( \int_0^1 \int_x^1 \cos(y^2) \, dy \, dx \) by first reversing the order of integration.

The region \( D \) over which the integration takes place is shown on the right. It can be described in two ways.

\[
0 \leq x \leq 1 \quad \text{or} \quad 0 \leq x \leq 1
\]

Using the second description, the integral iterates, and evaluates, as follows.

\[
\int_0^1 \int_y^0 \cos(y^2) \, dx \, dy = \int_0^1 \left[ x \cos(y^2) \right]_{x=0}^{x=y} \, dy = \int_0^1 y \cos(y^2) \, dy = \frac{1}{2} \sin(y^2) \bigg|_0^1 = \frac{1}{2} \sin(1)
\]

b) Find the volume and the centroid of the solid \( E \) that lies above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = 1 \). Hint. By symmetry, \( \bar{x} = \bar{y} = 0 \).

The solid region \( E \) is shown on the right. It is described in spherical coordinates like this.

\[
0 \leq \theta \leq 2\pi \\
0 \leq \phi \leq \pi/4 \\
0 \leq \rho \leq 1
\]

Therefore, the volume can be found as follows.

\[
V(E) = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin(\phi) \, d\phi \, d\theta
\]

\[
= \frac{2\pi}{3} \left[ -\cos(\phi) \right]_0^{\pi/4} = \frac{2\pi(1 - \sqrt{2}/2)}{3} = \frac{\pi(2 - \sqrt{2})}{3}
\]

The z-coordinate of the centroid of \( E \) is \( \bar{z} = \frac{1}{V(E)} \iiint_E z \, dV \). Using spherical coordinates,

\[
\iiint_E z \, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho \cos(\phi) \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = \frac{1}{4} \int_0^{2\pi} \int_0^{\pi/4} \cos(\phi) \sin(\phi) \, d\phi \, d\theta
\]

\[
= \frac{2\pi}{4} \left[ \frac{1}{2} \sin^2(\phi) \right]_0^{\pi/4} = \frac{\pi}{8} \cdot \frac{1}{4} = \pi/8.
\]

Therefore,

\[
\bar{z} = \frac{\pi/8}{\pi(2 - \sqrt{2})/3} = \frac{3}{8(2 - \sqrt{2})} \approx 0.6402.
\]
3. Do both parts.

a) Approximate the integral \[ \int_{-1}^{1} \int_{-1}^{1} e^{-(x^2+y^2)} \, dy \, dx \] using a Midpoint Rule Riemann sum with 4 subrectangles \((n = 2)\). Compare this to the approximate value obtained using your calculator (or Maple).

The domain of integration splits into 4 squares, as shown on the right. The centers of the squares are \((\pm 1/2, \pm 1/2)\) where the integrand has the value \(e^{-1/2} = 1/\sqrt{e}\).

Since each square has area 1, the Midpoint Rule approximation is

\[ 4 \cdot \frac{1}{\sqrt{e}} \approx 2.426. \]

According to Maple,

\[ \int_{-1}^{1} \int_{-1}^{1} e^{-(x^2+y^2)} \, dy \, dx \approx 2.231. \]

b) Do both parts.

i. Show that the vector field \( \mathbf{F} = (y, x+2y) \) is conservative and find a potential function.

Since \( P = y \) and \( Q = x + 2y \), \( P_y = 1 \) and \( Q_x = 1 \) so the field is exact.

A potential function \( f \) must have the property that

\[ f_x = y \quad \text{and} \quad f_y = x + 2y. \]

Therefore,

\[ f(x,y) = \int y \, dx = xy + g(y) \quad \text{and} \quad f(x,y) = \int x + 2y \, dy = xy + y^2 + h(x). \]

Consequently, \( f(x,y) = xy + y^2 \) can be used as a potential function for \( \mathbf{F} \).

ii. Evaluate the line integral \( \int_C y \, dx + (x + 2y) \, dy \) where the path \( C \) is the semicircle \((x - 1)^2 + (y - 1)^2 = 1, \, y \geq 1, \) starting at the point \((0,1)\) and ending at the point \((2,1)\).

According to the Fundamental Theorem of Line Integrals,

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x,y)\bigg|_{(0,1)}^{(2,1)} = f(2,1) - f(0,1) = 2 + 1 - 1 = 2. \]