1. (Chapter 12.8, Exercise 35) The function defined by the power series \[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n! (n + 1)! 2^{2n+1} n^2} \] is called the Bessel Function of the First Kind of Order 1. In textbooks this function is denoted \( J_1(x) \). In Maple it is named \( \text{BesselJ}(1, x) \).

a) Use the Ratio Test to verify that the domain of this Bessel function is the entire \( x \) axis.

b) Graph several partial sums on the same set of axes. The easiest way to do this is to make the following definition

\[ S(N, x) := \sum_{n=0}^{N} \frac{(-1)^n x^{2n+1}}{n! (n + 1)! 2^{2n+1} n^2} \]

Then put several partial sums into a list inside a plot procedure something like this.

\[ \text{plot}(\{S(7, x), S(12, x), S(22, x), S(28, x)\}, x = 0..30, \text{view} = 0.5..0.8) \]

c) Read the Help page for Bessel Functions and make a new graph showing the partial sums that you plotted in part b along with the plot of \( J_1(x) \). Enter it in Maple as \( \text{BesselJ}(1, x) \). Comment on how the partial sums approximate the Bessel function.

2. It is known that the equation \( \sum_{n=0}^{\infty} n \cdot x^n = \frac{x}{(x-1)^2} \) holds for \(-1 < x < 1\). In this exercise you will explore this equation by graphing partial sums \( S(N, x) = \sum_{n=0}^{N} n \cdot x^n \) of \( \sum_{n=0}^{\infty} n \cdot x^n \) for several values of \( N \).

a) Plot \( f(x) = x / (x-1)^2 \) for \(-0.5 < x < 0.5\). In the same window, first add the plot of \( S(2, x) \), then the plot of \( S(4, x) \), and finally \( S(6, x) \). You will notice the increasing accuracy as \( N \) increases.

b) Repeat part a but plot over the interval \(-0.8 < x < 0.5\). Notice that the convergence of \( S(N, x) \) is less rapid as \( x \) moves away from the center 0 of the interval of convergence. Replace the plot of \( S(8, x) \) with a plot of \( S(20, x) \). You will see that, for \( x \) in the interval of convergence but near an endpoint, \( S(N, x) \) can be used to accurately approximate \( f(x) \), but a large value of \( N \) may be required.

c) Repeat part a but plot over the interval \( 0.5 < x < 0.9 \). Replace the plot of \( S(8, x) \) with a plot of \( S(25, x) \).

3. The function \( p \to \sum_{n=1}^{\infty} \frac{1}{n^p} \) is called the Riemann Zeta Function: \( \zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} \).

a) Maple has the zeta function built in. Test it by typing "zeta", pressing the escape key, then return, and evaluate \( \zeta(2) \). You should get \( \frac{\pi^2}{6} \). You may remember that in class one day I mentioned that

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \].
b) Use Maple to find \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) and then evaluate \( \zeta(4) \) to a floating point number using \textit{evalf}.

c) Use Maple to find the exact value of \( \sum_{n=1}^{\infty} \frac{1}{n^3} \). Approximate the exact value by applying \textit{evalf}.

d) Leonard Euler was able to determine rational numbers \( q_k \) such that \( \zeta(2k) = q_k \pi^k \) for each positive integer \( k \). Use Maple to find \( q_3 \), \( q_4 \), and \( q_5 \).

e) Very little is known about \( \zeta(n) \) for \( n \) an odd integer. For example, it was not until 1978 that the sum \( \zeta(3) \) was shown to be irrational. Moreover, no one knows the exact value of \( \zeta(3) \) nor the exact sum of any \( p \)-series where \( p \) is odd.

i. Obtain the graph of the zeta function on the interval \( 1 < p < 20 \). Get a nicer picture using the following view equation: \textit{view} = \([1 .. 20, 0 .. 6]\).

ii. Look for zeros of the zeta function in the complex plane by experimenting with 3d plots of the expression \( |\zeta(x + I \cdot y)| \) for \( x = a .. b, y = c .. d \) with various choices for \( a, b, c, d \), and various views (i.e. vertical ranges). The symbol \( I \) stands for the complex number \( \sqrt{-1} \). Start with the following entry. Prizes will be given for the best picture and the number of zeros that you can find.

\[
\text{plot3d} (|\zeta(x + I \cdot y)|, x = 0 .. 2, y = 0 .. 10, \textit{view} = 0 .. 3, \textit{style} = \text{patchcontour}, \textit{orientation} = [-40, 70], \textit{axes} = \text{normal})
\]

\begin{center}
\textbf{The Riemann Hypothesis}
\end{center}

Currently there is a prize of $1,000,000 to the first person to settle the so-called \textit{Riemann Hypothesis}. This is a statement about the location of the zeros of the Riemann Zeta Function. You can read about it here.

http://en.wikipedia.org/wiki/Riemann_hypothesis

Or you can watch a movie here.

http://www.youtube.com/watch?v=a8AO_Q8ANI4