15.2-3: Limits and Continuity; Partial Derivatives

These questions are meant to guide your initial study of the material prior to our discussion in class. For the non-computational questions, please give short answers for each: 2-3 sentences or a small paragraph.

Since these questions form an initial investigations of these ideas, you are not expected to always be able to answer these question fully. They are intended to raise key topics and questions that can be discussed in more depth and detail in subsequent classes. Ideally, you should go through the chapter prior to class and try a couple of problems. Then once we’ve had a deeper discussion in class, revisit the chapter (and these questions) and then do many problems covering that material. The reading provides the background which will guide you through the work. The homework problems then provide the experience in implementing the concepts and cover some the variability which can occur in problems for that material.

Caveat: These questions are not intended to be a comprehensive summary of all the ideas in this section; they offer a selected subset of those ideas.

The $\delta$-$\epsilon$ definition of the limit, Definition 1 in Section 15.2 on pg. 907, is a rigorous definition of the limit which is very useful once mastered, but it may be hard to follow at first. (You are not responsible for using this definition in this class.) Both the sentence above it and the paragraph near the end of the next page (just before the box above Example 1) capture the core idea of the limit in less rigorous language. Figures 1 and 2 give a graphical representation of the concept.

1. **Limits for functions of 2 variables: paths and full 2D limits.**

   Consider the function $f(x, y)$ given at the beginning of section 15.2, pg. 906.

   (a) Using Table 1, estimate the limit of $f(x, y)$ as $(x, y)$ approaches $(0, 0)$ along two paths: (i) $y = 0, x > 0$; (ii) $y = x, x > 0$.

   (b) What does the behavior of $f(x, y)$ near $(0, 0)$ in Table 1 suggest about $\lim_{(x,y) \to (0,0)} f(x, y)$.

   (c) Repeat using $g(x, y)$ and Table 2.

   (d) How are the (1D) limits along paths related the full (2D) limit?

2. **Using paths to show a limit does not exist. Behavior near such a point.**

   Read Example 2, pg. 908.

   (a) What are the limits along the following 6 paths? (Compare to Figure 5.)

   $(i) y = 0, x > 0$; $(ii) y = 0, x < 0$; $(iii) x = 0, y > 0$;

   $(iv) x = 0, y < 0$; $(v) y = x, x > 0$; $(vi) y = x, x < 0$
(b) Find the limit of \( f(x, y) \) as \((x, y)\) approaches \((0, 0)\) along the path \( y = -x, x > 0 \).

(c) Compare to the graph of \( f(x, y) \) shown in Figure 6.

(d) For each of the path you just used, find the image of the path on the surface.

(e) From the graph and without further calculation, estimate the limit of \( f(x, y) \) as \((x, y)\) approaches \((0, 0)\) along the path \( y = x/2, x > 0 \).

3. **Continuity**

Read examples 6, 7, 8 on pg. 911. The function in Example 6 is discontinuous at \((0, 0)\) simply because it is not defined there; it’s graph has a ”pinhole” there. For the functions in Examples 7 and 8, we have filled in the pinhole by assigning a value to \( f(x, y) \) at \((x, y) = (0, 0)\).

(a) Why is the resulting function in Example 8 continuous at the origin while the function in Example 7 is still discontinouus there?

(b) Back to Example 6: is it possible to assign a value for \((x, y) = (0, 0)\) such that the resulting function is now continuous at the origin? Refer to Example 1, Section 15.2, pg. 908 if needed.

4. **Definition of the Partial Derivatives: Section 15.3.**

Consider Table 1, pg. 914, and the discussion following it up to the definition of the partial derivatives as a limits, Definition 2, pg. 915 and Definition 3 on pg. 916.

(a) How is the partial derivative of \( f(x, y) \) with respect to \( x \) at a point \((a, b)\), \( \frac{\partial f}{\partial x} (a, b) \), related to the (1D) derivative of a function in \( x \) only?

(b) What is that function of \( x \) in terms of \( f, a \) and \( b \)?

(c) How is this related to taking vertical slices and the resulting traces?

See Figures 4 and 5, pg. 918.

5. Write down a question you have relating to this material (and consider asking it in class).