16.8: Triple Integrals in Spherical Coordinates

These questions are meant to guide your initial study of the material *prior* to our discussion in class. For the non-computational questions, please give short answers for each: 2-3 sentences or a small paragraph.

Since these questions form an initial investigations of these ideas, you are not expected to always be able to answer these question fully. They are intended to raise key topics and questions that can be discussed in more depth and detail in subsequent classes. Ideally, you should go through the chapter prior to class and try a couple of problems. Then once we’ve had a deeper discussion in class, revisit the chapter (and these questions) and then do many problems covering that material. The reading provides the background which will guide you through the work. The homework problems then provide the experience in implementing the concepts and cover some the variability which can occur in problems for that material.

Caveat: These questions are not intended to be a comprehensive summary of all the ideas in this section; they offer a selected subset of those ideas.

1. **Coordinate Surfaces:**

   Figures 2-4 shows examples of the the constant-$\rho$, constant-$\theta$ and constant-$\phi$ “coordinate surfaces” in spherical coordinates. Note: $\phi = c$ is a half-cone, not a full cone.

   (a) Sketch and algebraically define a solid which is bounded solely by spherical coordinate surfaces.

   (b) Can you define an “ice-cream cone”-like shape using only spherical coordinate surfaces?

2. **Changing Coordinate Systems:**

   Equations 1 and 2 state the coordinate transformations between Cartesian and spherical coordinates. Examples 3 and 4 demonstrate how to use them to change the definition of a surface from Cartesian to spherical coordinates.

   (a) Find the equation for the paraboloid, $x^2 + y^2 + z = 1$, in spherical coordinates. Compare this result to the equation in cylindrical coordinates given in Sec. 16.7, Example 3, p.1039. Which representation is simpler, spherical or cylindrical?

   (b) Using Equations 1 and 2, find an equation in Cartesian coordinates for the surface $\rho = 2 \cos \phi$. (Hint: multiply by $\rho$ first.)

   (c) Sketch the surface in part (b). Hint: move all variables to one side of the equation and complete the square.

   (d) Starting with the Cartesian equation from part (b), find an equation for $\rho = 2 \cos \phi$ in cylindrical coordinates. Which is simpler, spherical or cylindrical?
3. Volume Elements and the Differential $dV$:

Figures 7 and 8 shows a “volume element” in spherical coordinates. Consider a similar-shaped solid with positive, but small, $\Delta \rho$, $\Delta \theta$ and $\Delta \phi$.

(a) Does the volume of this solid change (for fixed $\Delta \rho$, $\Delta \theta$ and $\Delta \phi$) when it is moved radially, i.e., a change in $\rho$ while holding $\theta$ and $\phi$ fixed?

(b) What about when moving it in $\theta$ holding $\rho$ and $\phi$ fixed, i.e., rotation about the $z$-axis?

(c) What about moving it in $\phi$ only, i.e., change the angle relative to the $z$-axis? This one might be harder to see. Consider one of the edges in Figure 8 associated with $\Delta \theta$; what happens to its length, $\rho \sin \phi \Delta \theta$ as $\phi$ shrinks to 0?

(d) Is this consistent with the definition of the volume element in Formula 3, i.e., $dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$?


Formula 3 defines a triple integral for $E$ when $E$ is a spherical “box” or wedge; i.e., purely numeric bounds. However, the solid region $E$ defined in example 4 is not a spherical wedge; but it is $\rho$-simple. Furthermore, the domain of the middle and outer variables, $\phi$ and $\theta$, are purely numeric. The volume element (and order of integration) is given by $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$; Figure 11 demonstrates the order in which all the points in the $E$ are swept through.

Since the bounds for the outer variables, $\theta$ and $\phi$, are both numeric, their order can be swapped, i.e., $dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$. The first sketch in Figure 11 remains the same.

(a) What would the next two sketches look like using this order of integration?

(b) What is the new iterated integral for the volume, $V(E)$?

5. Create a question about this (or any of the preceding) material.