Math 142 (Camp)

Worksheet: Integration by Parts

\[ \int u \, dv = uv - \int v \, du \]

Split your original integrand into two parts \( u \) and \( dv \) such that

- the antiderivative \( v = \int dv \) is easy to find, and
- the new integral, \( \int v \, du \), is easier to determine than the original integral, \( \int u \, dv \).

Problems:

1. Use integration by parts to solve the following integrals

   (a) \[ \int x \ln x \, dx \]
   
   \[ \text{Follow the hint above.} \]
   
   \[ \text{Choose } u = \ln x \quad \Rightarrow \quad du = \frac{1}{x} \, dx \]
   
   \[ \text{dv} = x \, dx \quad \Rightarrow \quad v = \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx \]
   
   \[ \Rightarrow \int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C \]

   Let's try the problem again, but let \( u = x \quad \Rightarrow \quad \]
   
   \[ du = dx \quad \Rightarrow \quad v = \int \ln x \, dx = \frac{x}{2} \ln x - \frac{x}{2} \quad \text{goal} \]

   (b) \[ \int x^2 e^{-x} \, dx \]

   Hint: sometimes you need to use IBP more than once.

   \[ \text{Let's try } u = x^2 \quad \Rightarrow \quad \]
   
   \[ du = 2x \, dx \quad \Rightarrow \quad v = -e^{-x} \quad (= \int e^{-x} \, dx) \]

   \[ \Rightarrow \int x^2 e^{-x} \, dx \]

   \[ \Rightarrow \int x e^{-x} \, dx \quad \text{(IBP again)} \]

   \[ \Rightarrow \int e^{-x} \, dx \quad \text{let } u = x \quad \Rightarrow \quad \]
   
   \[ du = dx \quad \Rightarrow \quad v = -e^{-x} \]

   \[ \Rightarrow \int x e^{-x} \, dx = -xe^{-x} + \int e^{-x} \, dx \]

   \[ = -xe^{-x} - e^{-x} + C \]

   \[ = xe^{-x} - e^{-x} + C \]

   \[ \text{Use in previous work} \]

   \[ \text{Note: still differentials.} \]

   \[ \text{We obtained the first polynomial. If we had let } u = e^{-x}, \text{ we would have gotten the \textit{final} result, and ended up back at the beginning!} \]
2. We're going to use integration by parts to determine the integral

\[ I = \int e^x \sin x \, dx. \]

(a) Perform integration by parts once, letting \( u = \sin x \) and \( dv = e^x \, dx \).

\[ \Rightarrow I = \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx \]

\[ \quad \text{with} \quad \begin{align*}
  u &= \sin x & dv &= e^x \, dx \\
  \frac{du}{dx} &= \cos x & v &= e^x 
\end{align*} \]

\[ = e^x \sin x - \int e^x \cos x \, dx \]

(b) Perform integration by parts again on the resulting new integral, letting \( u = \cos x \) and keeping \( dv = e^x \, dx \). (Don't reverse them in the second IBP, you'll just end up undoing the first IBP.)

\[ \int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx \]

\[ \quad \text{with} \quad \begin{align*}
  u &= \cos x & dv &= e^x \, dx \\
  \frac{du}{dx} &= -\sin x & v &= e^x 
\end{align*} \]

\[ \Rightarrow \quad \int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \]

\[ \Rightarrow \quad 2I = e^x \sin x - e^x \cos x \]

\[ \Rightarrow \quad I = \frac{1}{2} \left( e^x \sin x - e^x \cos x \right) + C \]

(c) Notice that the original integral, \( I \), has reappeared. You should have an equation which looks like \( I = h(x) + cI \) for some function \( h(x) \) and some constant, \( c \). Solve this for equation for \( I \), our desired integral. (Don't forget the constant of integration.)

\[ \Rightarrow \quad I = h(x) + cI \]

\[ \Rightarrow \quad 2I = e^x \sin x - e^x \cos x \]

\[ \Rightarrow \quad I = \frac{1}{2} \left( e^x \sin x - e^x \cos x \right) + C \]

\[ \text{with} \quad \begin{align*}
  h(x) &= e^x \sin x - e^x \cos x \\
  c &= -1 
\end{align*} \]

We've done many problems as long as \( h(x) \) is a \( C \), \( c \neq 1 \).

(d) Confirm your result by differentiating it.

You should be able to show that \( \frac{dI}{dx} = e^x \sin x \) (original integral)

\[ \frac{d}{dx} \left[ \frac{1}{2} (e^x \sin x - e^x \cos x) \right] + \frac{1}{2} \frac{d}{dx} \left[ e^x (\sin x - \cos x) \right] + C = \frac{1}{2} e^x (\sin x + \cos x) + e^x (\sin x - \cos x) \]

\[ = \frac{1}{2} e^x (2 \sin x + 2 \cos x) = \frac{1}{2} e^x (2 \sin x + 2 \cos x) \]

\[ = e^x \sin x \]

You can confirm the previous 2 integrals in a similar way.

\[ \frac{d}{dx} \left[ \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C \right] = \frac{1}{2} x \cdot \frac{1}{x} + \frac{1}{2} (2x) \ln x - \frac{1}{2} x + 0 = \frac{1}{2} x - \frac{1}{2} x = \frac{1}{2} x = x \ln x \]