Math 241 (Camp)

A triple integral in non-Cartesian coordinates

Let $E$ be the region inside the (offset) sphere and above the (half) cone defined by

$$x^2 + y^2 + z^2 = z \quad \text{and} \quad z = 2\sqrt{x^2 + y^2}$$

1. Rewrite the equation for the sphere in standard form:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

2. Sketch $E$ and find algebraic representations for it in both cylindrical and spherical coordinates. Before actually determining the two representations, which do you think should be simpler to work with?

Hints: (i) In spherical coordinates, change the equation for the sphere to spherical coordinates (via the coordinate transformation) and solve for $\rho(\phi, \theta)$. Then find the bounds for $\phi$ for the cone. (ii) In cylindrical coordinates, change equation for the sphere to cylindrical coordinates and solve for $z(r, \theta)$. Then find the intersection between the sphere and the cone to find the projected domain in $r-\theta$ (i.e., in the $x$-$y$ plane).

3. Using both the coordinate systems above, setup triple integrals for the following 2 problems:

(a) the volume of $E$, and

(b) $\iiint_E y^2 z^2 \, dV$

4. Evaluate all of the above integrals.