Directional Derivatives and the Gradient

You are standing on a hill whose elevations are given by \( z = f(x, y) = x^2 y^3 \) where \( x, y, z \) are in meters; it's a strange-shaped hill. You are standing at the point \((-1, 2, 8)\) on the hill. Hint: you might want to find the gradient at your location before answering the following questions.

\[
\nabla f = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right> = \left< 2xy^3, 3x^2y^2 \right>
\]

Note: "location" is the domain \((x, y) = (-1, 2)\); \((-1, 2, 8)\) is the point \((x, y, f(x, y))\) on the surface.

\[
\nabla f(-1, 2) = \left< -16, 12 \right>
\]

1. What is the slope you are facing if you are looking due east?
   Are you facing uphill or downhill?

   Two (related) ways to do this:

   i) East is positive \(x\) direction

   \[
   \text{Slope} = \left. \frac{dy}{dx} \right|_{(1,2)} = 2xy^3 \bigg|_{(-1,2)} = -16
   \]

   ii) East is the direction \(\hat{e} = \langle 1, 0 \rangle\)

   Directional Derivative is

   \[
   \partial_{\hat{e}} f = \nabla f \cdot \hat{e} = \left< -16, 12 \right> \cdot \left< 1, 0 \right> = -16
   \]

   Either way, \(16 < 0\) \(\Rightarrow\) you are facing downhill.

2. If you are facing northwest, are you facing uphill or downhill?
   What is the slope in that direction?

   NW direction \(\Rightarrow\) \(\hat{e} = \langle -1, 1 \rangle\)

   Need unit vector \(\hat{u} = \frac{\hat{e}}{||\hat{e}||} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle\)

   \[
   \partial_{\hat{u}} f = \nabla f \bigg|_{(-1,2)} \cdot \hat{u} = \left< -16, 12 \right> \cdot \left< -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right> = \frac{16}{\sqrt{2}} + \frac{12}{\sqrt{2}} = \frac{28}{\sqrt{2}} = 14\sqrt{2} > 0
   \]

   Same location, new direction, same gradient.
3. In what direction should you face to be looking at the steepest uphill? What is the slope in that direction?

\[ \vec{u} = \nabla f = \langle -16, 12 \rangle \]

slope is \[ ||\nabla f|| = \sqrt{(-16)^2 + 12^2} = \sqrt{256 + 144} = 4 \sqrt{20} = 4 \cdot 5 = 20 \]

Note: slope \( \geq 0 \) as expected.

The direction can be rescaled (normalized) by multiplying by some positive number. e.g., \( \vec{u} = \langle -4, 3 \rangle \) or \( \vec{v} = \langle -\frac{4}{5}, \frac{3}{5} \rangle \) are valid answers for the direction, but slope requires the actual gradient (unscaled).

\[ ||\nabla f|| = ||\langle -16, 12 \rangle|| \]

4. In what direction(s) should you start to head if you do not want to change your elevation?

We want to stay on a level curve \( \Rightarrow \) vertically head perpendicular \( \bot \) to the gradient.

Note: \( \langle a, b \rangle \bot \langle -b, a \rangle \) and \( \langle a, b \rangle \bot \langle b, -a \rangle \)

since \( \langle a, b \rangle \cdot \langle -b, a \rangle = -ab + ab = 0 \)

so we want a direction \( \vec{u} \) such that \( \nabla f \cdot \vec{u} = 0 \)

\( \nabla f = \langle -16, 12 \rangle \), so \( \vec{u} = \langle 12, -16 \rangle \) or \( \vec{u} = \langle -12, -16 \rangle \)

check: \( \vec{u} \cdot f = \nabla f \cdot \vec{u} = \langle -16, 12 \rangle \cdot \frac{\langle 12, -16 \rangle}{20} = 0 \)

\[ \vec{u} = \nabla f \]

\( m = \text{scale} \)