32) \[ I = \int_0^1 \int_0^{1-x} \int_0^{-x^2} f(x, y, z) \, dz \, dy \, dx \]

a) \( D \) in projection is \( x \)-\( z \) plane

\[ 0 \leq x \leq 1 \quad \text{outer} \]
\[ 0 \leq y \leq 1-x^2 \quad \text{middle} \quad \text{inner} \]
\[ 0 \leq z \leq 1-x^2 \]

\[ D: \{ (x,y,z) \mid 0 \leq x \leq 1, \ 0 \leq z \leq 1-x^2 \} \]

- Keep \( y \) as innermost integral, exchange order in \( D \) to get

\[ D: \{ \begin{cases} 0 \leq z \leq 1 \\ 0 \leq x \leq \sqrt{1-z} \\ 0 \leq y \leq 1-x \end{cases} \} \]

\[ I = \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx \]

b) \( D \) as projection to \( x \)-\( y \) plane

- In original, exchange order of \( y \) \& \( z \) integrals, i.e., do \( f \) \( x \)-\( y \), \( x \)-\( z \)

\[ \Rightarrow \{ \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1-x \quad \text{inner} \\ 0 \leq z \leq 1-x^2 \quad \text{outer} \end{cases} \} \]

\[ I = \int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dz \, dy \, dx \]

(Note: \( x \) \& \( z \) outer, bounds of \( y \) \& \( z \) are independent of each other \( \Rightarrow \) we can simply exchange their order)

- Keeping \( z \) as innermost, exchange order of \( x \)-\( y \) in \( D \)

\[ 0 \leq y \leq 1 \]
\[ 0 \leq x \leq 1-y \]
\[ 0 \leq z \leq 1-x^2 \]

\[ I = \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x, y, z) \, dz \, dx \, dy \]
We've done both possibilities for \( y \) as a constant and for \( z \) as a constant in \( y \).

(i.e., \( y \) is projection into \( x-y \) and \( x-y \) planes, respectively.)

Now, we'll try to projecting into \( y-z \) plane to \( 0 \), i.e. \( x \) as invariant.

\[ D: \begin{aligned}
0 &\leq z \\
0 &\leq x \\
0 &\leq z = 1 - y
\end{aligned} \]

and lower bound for \( x \) is \( x = 0 \).

Unfortunately, the upper bound for \( x \) is a piecewise function.

\[ x \begin{aligned}
0 &\leq x \leq \sqrt{y} \\
0 &\leq x \leq 1 - y
\end{aligned} \]

We need to find the curve in the \( y-z \) plane which separates these regions.

It's the projection of the intersection of \( z = 1 - x^2 \) and \( y = 1 - x \) into the \( y-z \) plane.

Intersection occurs when \( z = 1 - x^2 \) holds true

\[ y = 1 - x \]

\[ y = 1 - x \Rightarrow x = 1 - y \]

\[ z = 1 - (1-y)^2 \Rightarrow z = 2y - y^2 \]

so \( D \) is the unit square in \( y-z \) plane divided along \( z = 2y - y^2 \).

\[ \left\{ \begin{array}{c}
0 \leq y \leq 1 \\
0 \leq z \leq 2y - y^2 \\
0 \leq x \leq 1 - y
\end{array} \right\} 
\]

\[ \Rightarrow I = \int_{y=0}^{y=1} \int_{z=0}^{z=2y-y^2} f(x,y,z) \, dx \, dz \, dy 
+ \int_{y=1}^{y=\sqrt{1+y^2}} \int_{z=0}^{z=2y-y^2} f(x,y,z) \, dx \, dz \, dy 
\]

= summing order of \( x \) then \( y \)

\[ \left\{ \begin{array}{c}
0 \leq z \leq 1 - \sqrt{1+y^2} \\
0 \leq y \leq 1 - \sqrt{1+y^2} \\
0 \leq x \leq \sqrt{1+y^2}
\end{array} \right\} 
\]

\[ I = \int_{y=0}^{y=1} \int_{z=0}^{z=2y-y^2} f(x,y,z) \, dx \, dz \, dy 
+ \int_{y=1}^{y=\sqrt{1+y^2}} \int_{z=0}^{z=2y-y^2} f(x,y,z) \, dx \, dz \, dy 
\]