A vector field, $\mathbf{F}$, is a gradient vector field if $\mathbf{F} = \nabla f$ for some scalar field, $f$; $f$ is often called a potential function of $\mathbf{F}$. Gradient vector fields are very important in both mathematical and physical problems. For example, the fundamental theorem of calculus can be extended to line integrals of gradient vector fields.

**Theorem 1:** The Fundamental Theorem of Calculus for Line Integrals: Suppose that $f$ is a differentiable scalar function and $C$ is a piecewise smooth curve, then

$$ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(r(b)) - f(r(a)) $$

where $\mathbf{F} = \nabla f$ and $C$ is parameterized by $\mathbf{r}(t)$, $a \leq t \leq b$.

We need to be able to determine when a vector field can be written as a gradient. We will first discuss this for 2D vector fields, $\mathbf{F}(x, y)$, then extend this to 3D vector fields, $\mathbf{F}(x, y, z)$.

**Theorem 2:** Conservative Fields in $\mathbb{R}^2$: Let $D$ be an open simply-connect region in $\mathbb{R}^2$ and let $\mathbf{F} = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$ be a differentiable field defined on $D$. The following conditions on $\mathbf{F}$ are all equivalent:

i. For any oriented simple closed curve $C$ in $D$, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

ii. For any two oriented simple curves, $C_1$ and $C_2$, in $D$ that have the same endpoints,

$$ \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}; $$

when this holds, the line integral is called **path independent**.

iii. $\mathbf{F}$ is the gradient of some function $f$ in $D$; i.e., $\mathbf{F} = \nabla f$ in $D$.

iv. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in $D$.

A vector field satisfying one (and, therefore, all) of the conditions (i)-(iv) is called a **conservative vector field**.

**Proof** We shall prove Theorem 2 cyclically, i.e., we will separately prove 4 implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). If we then assume any one of the four statements is true, we can conclude that any of the remaining statements must then be true; hence, the four statements are all equivalent.
1. (i) ⇒ (ii): Let \( C_1 \) and \( C_2 \) be any two oriented simple curves in \( D \) with the same endpoints. Let \(-C_2\) denote the curve \( C_2 \) with a reversed orientation. Then \( C = C_1 \cup (-C_2) \) is a simple closed curve in \( D \). By hypothesis (1), \( \oint_C F \cdot dr = 0 \); therefore,

\[
\oint_C F \cdot dr = \oint_{C_1} F \cdot dr + \oint_{-C_2} F \cdot dr = \oint_{C_1} F \cdot dr - \oint_{C_2} F \cdot dr = 0 \implies \int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr.
\]

2. (ii) ⇒ (iii): Assume \( C \) is an oriented simple curve in \( D \) connecting a fixed point \((a, b)\) in \( D \) to any arbitrary point \((x, y)\) in \( D \). Let \( f(x, y) = \int_C F \cdot dr \). By hypothesis (2), \( f(x, y) \) is independent of \( C \). Therefore, we can define a new path \( C_1 \), consisting of the two line segments connecting \((a, b)\) to \((x, b)\) and \((x, b)\) to \((x, y)\), such that

\[
f(x, y) = \int_C F \cdot dr = \int_{C_1} F \cdot dr = \int_a^x P(t, b) \, dt + \int_b^y Q(x, t) \, dt
\]

\[
\implies \frac{\partial f}{\partial y}(x, y) = Q(x, y).
\]

Similarly, we can define another path, \( C_2 \), connecting \((a, b)\) to \((x, y)\) consisting of the two line segments connecting \((a, b)\) to \((a, y)\) and \((a, y)\) to \((x, y)\). Then, invoking Clairaut’s Theorem, we get

\[
f(x, y) = \int_C F \cdot dr = \int_{C_2} F \cdot dr = \int_b^y Q(a, t) \, dt + \int_a^x P(t, y) \, dt
\]

\[
\implies \frac{\partial f}{\partial x}(x, y) = P(x, y).
\]

Therefore \( F = P\hat{i} + Q\hat{j} = \frac{\partial f}{\partial y}\hat{i} + \frac{\partial f}{\partial x}\hat{j} = \nabla f \).

3. (iii) ⇒ (iv): Assume \( F = \nabla f \) in \( D \) for some differentiable scalar field, \( f(x, y) \), where \( F = P\hat{i} + Q\hat{j} \). Then

\[
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \, \partial x} = \frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial Q}{\partial x}.
\]

4. (iv) ⇒ (i): Let \( C \) be any simple closed curve in \( D \) and define \( D_1 \) to be the simply connected region contained by \( C \), i.e., \( C = \partial D_1 \). Since \( D \) is simply connected,

\[1\]You might find useful to sketch \( C_1 \).
\[ D_1 \subset D; \text{ therefore, by hypothesis (4), } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial y} \text{ for all points in } D_1. \]

Invoking Green's Theorem, we get
\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P \, dx + Q \, dy = \iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = 0.
\]

The theorem defining conservative fields in \( \mathbb{R}^3 \) is quite similar. The two differences are the use of the curl of \( \mathbf{F} \) in the fourth statement and the less restrictive requirements on the domain in the hypothesis.

**Theorem 3: Conservative Fields in \( \mathbb{R}^3 \):** Let \( \mathbf{F} = P(x, y, z) \hat{i} + Q(x, y, z) \hat{j} + R(x, y, z) \hat{k} \) be a differentiable field defined on \( \mathbb{R}^3 \), except possibly for a finite number of points. The following conditions on \( \mathbf{F} \) are all equivalent:

i. For any oriented simple closed curve \( C \), \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \).

ii. **Path Independence:**

   For any two oriented simple curves, \( C_1 \) and \( C_2 \), that have the same endpoints,
   \[
   \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.
   \]

iii. **Existence of a potential function:**

   \( \mathbf{F} \) is the gradient of some function \( f \); i.e., \( \mathbf{F} = \nabla f \).

iv. \( \nabla \times \mathbf{F} = 0 \).

A vector field satisfying one (and, therefore, all) of the conditions (i)-(iv) is called a **conservative vector field**.

Theorem 3 can be proven in a similar way to the proof given for Theorem 2. Key differences:

- (ii) \( \Rightarrow \) (iii): use 3 different curves, each of which consists of 3 line segments parallel to the coordinate axes.

- (iii) \( \Rightarrow \) (iv): use the vector identity, \( \nabla \times \nabla f = 0 \).

- (iv) \( \Rightarrow \) (iv): invoke Stokes' Theorem (instead of Green's Theorem) for an oriented surface \( S \) whose boundary is \( \partial S = C \).


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\[ \text{This statement creates the requirement that } D \text{ be a simply connected region in } \mathbb{R}^2; \text{ otherwise, } D_1 \text{ might not be a subset of } D \text{ and the equality might not hold for all points in } D_1. \]