Problem 1. Let $X$ be the set of all continuous real valued functions on $[0,1]$, and let $\rho : X \times X \to \mathbb{R}$ be the function $\rho(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|$.

(1) Prove that $(X, \rho)$ is a metric space.

Solution. We are required to show that:

(a) for all $f, g \in X$, $\rho(f,g) = \rho(g,f)$,
(b) $\rho(f,g) = 0$ if and only if $f = g$, and
(c) for all $f, g, h \in X$, $\rho(f,g) \leq \rho(f,h) + \rho(g,h)$

The first fact is obvious since

$$\rho(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)| = \sup_{t \in [0,1]} |g(t) - f(t)| = \rho(g,f).$$

The second fact is similarly immediate since $\rho(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)| = 0$ can only occur if $f(t) = g(t)$ for all $t \in [0,1]$, that is, if $f = g$, and obviously $f = g$ implies that $\rho(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)| = \sup_{t \in [0,1]} 0 = 0$. Finally we observe that

$$\rho(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)| = \sup_{t \in [0,1]} |f(t) - h(t) + h(t) - g(t)| \leq \sup_{t \in [0,1]} (|f(t) - h(t)| + |h(t) - g(t)|)$$

where the last inequality holds since

$$|f(t) - h(t) + h(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|$$

for each $t \in [0,1]$ by the usual triangle inequality. We finish by observing that

$$\sup_{t \in [0,1]} (|f(t) - h(t)| + |h(t) - g(t)|) \leq \sup_{t \in [0,1]} |f(t) - h(t)| + \sup_{t \in [0,1]} |h(t) - g(t)| = \rho(f,h) + \rho(h,g) = \rho(f,h) + \rho(g,h)$$

as required. \(\square\)

(2) Prove that the collection $P$ of polynomials with domains taken to be $[0,1]$ is dense in $X$.

Solution. We need to show that the closure of $P$ in $X$ is all of $X$. Since the closure of $X$ is the union of $P$ with all its accumulation points, it is enough to show that any $f \in X$ is an accumulation point of $P$. That is, it is enough to show that for $f \in X$ and $\epsilon > 0$, there is $p \in P$ such that $\rho(p,f) < \epsilon$. By the Weierstrass approximation theorem, there does exist a polynomial $p$ such that $|f(x) - p(x)| < \epsilon$ for all $x \in [0,1]$. It follows immediately that $\rho(p,f) = \sup_{t \in [0,1]} |f(t) - p(t)| < \epsilon$, and we are finished. \(\square\)

(3) Suppose that $f_j = \sum_{k=0}^{j} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ for $j = 1, 2, \ldots$. Show that $f_j \to \sin x$ as elements of $X$.

Solution. Let $\epsilon > 0$ be given. We need to demonstrate that there is $N > 0$ such that $n > N$ implies $\rho(\sin x, f_n) = \sup_{t \in [0,1]} |\sin x - f_n(x)| < \epsilon$. 
We know that $\sin x$ is infinitely differentiable on $(-2, 2)$, and hence by Taylor’s approximation theorem, (with $f(x) = \sin x$ for the sake of notation)

$$\sin x = \sum_{k=0}^{j} f^{(k)}(0) \frac{x^k}{k!} + \int_{0}^{x} f^{(j+1)}(t) \frac{(x-t)^{j+1}}{j!} dt.$$  

Of course,

$$\frac{d}{dx} \sin x \big|_{x=0} = \cos(0) = 1$$

$$\frac{d^2}{dx^2} \sin x \big|_{x=0} = -\sin(0) = 0$$

$$\frac{d^3}{dx^3} \sin x \big|_{x=0} = -\cos(0) = -1$$

$$\frac{d^4}{dx^4} \sin x \big|_{x=0} = \sin(0) = 0$$

$$\frac{d^5}{dx^5} \sin x \big|_{x=0} = \cos(0) = 1$$

$$\vdots$$

The pattern is that $\frac{d^k}{dx^k} \sin x \big|_{x=0} = \begin{cases} 1 & \text{if } k = 4\ell + 1 \\ -1 & \text{if } k = 4\ell + 3 \\ 0 & \text{if } k \text{ is even} \end{cases}$

and thus for each $j \in \mathbb{N},$

$$\sin x = \sum_{k=0}^{2j+1} f^{(k)}(0) \frac{x^k}{k!} + \int_{0}^{x} f^{(2j+2)}(t) \frac{(x-t)^{2j+1}}{(2j+1)!} dt$$

$$= \sum_{k=0}^{2j} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \int_{0}^{x} f^{(2j+2)}(t) \frac{(x-t)^{2j+1}}{(2j+1)!} dt = f_j(x) + \int_{0}^{x} f^{(2j+2)}(t) \frac{(x-t)^{2j+1}}{(2j+1)!} dt.$$  

This implies that

$$|\sin x - f_j(x)| = \left| \int_{0}^{x} f^{(2j+2)}(t) \frac{(x-t)^{2j+1}}{(2j+1)!} dt \right| \leq \int_{0}^{x} \left| f^{(2j+2)}(t) \right| \frac{(x-t)^{2j+1}}{(2j+1)!} dt.$$  

Since $f^{(2j+2)}(t) = \pm \sin t$ or $\pm \cos t$, it follows that $|f^{(2j+2)}(t)| \leq 1$ and hence

$$\int_{0}^{x} \left| f^{(2j+2)}(t) \right| \frac{(x-t)^{2j+1}}{(2j+1)!} dt \leq \int_{0}^{x} \frac{(x-t)^{2j+1}}{(2j+1)!} dt.$$  

But $(x-t)^{2j+1}$ is positive on $[0, x],$ so

$$\int_{0}^{x} \frac{(x-t)^{2j+1}}{(2j+1)!} dt = \int_{0}^{x} (x-t)^{2j+1} \left| \frac{(x-t)^{2j+1}}{(2j+1)!} \right| dt = \left| \frac{(0)^{2j+2}}{(2j+2)!} + (x)^{2j+2} \right| \leq \frac{1}{(2j+2)!}$$

since $x \in [0, 1].$

Now let $N$ be such that $(2N+2)! > \frac{1}{\epsilon}$. It follows that for all $n > N$,  

$$|\sin x - f_n(x)| \leq \frac{1}{(2N+2)!} \leq \frac{1}{(2N+2)!} < \epsilon,$$

and hence $\rho(\sin x, f_n) < \epsilon$ for all $n > N$. This completes the proof.  

$\Box$

(4) Suppose that $\mathcal{F}$ is a closed, infinite, equicontinuous, equibounded family of functions in $X$. Prove that $\mathcal{F}$ is compact in $X$.  

Proof. Suppose that Lemma 4. To show that countable unions of measurable sets are measurable. Repeat this construction, only this time from the point of view of intersections. Note: I expect you to redevelop the theory from the definitions, focusing on intersections where in class we considered unions.

Solution. We want to show that if \( f \) is a closed, equicontinuous, equibounded family, so by the previous problem, it is compact.

Solution. Since \( g \) is Riemann integrable on \([0, 1]\), it is bounded there, and since \( f(x) \leq g(x) \) for all \( x \in [0, 1], f \in \mathcal{F} \), it follows that \( \mathcal{F} \) is equibounded. Note also that \( \mathcal{F} \) is equicontinuous. To wit, for \( \epsilon > 0 \) given, let \( \delta = \epsilon^2 \). Then for all \( s, t \in [0, 1] \) such that \( |s - t| < \delta \), we have \( |f(s) - f(t)| < \sqrt{|s - t|} < \sqrt{\delta} = \sqrt{\epsilon^2} = \epsilon \) for all \( f \in \mathcal{F} \). It follows that \( \mathcal{F} \) is a closed, equicontinuous, equibounded family, so by the previous problem, it is compact. \( \square \)

Problem 2. In class we proved that countable unions of measurable sets are measurable. Repeat this construction, only this time from the point of view of intersections. Note: I expect you to redevelop the theory from the definitions, focusing on intersections where in class we considered unions.

Solution. We want to show that if \( E_j \) is measurable for all \( j \in \mathbb{N} \), then \( \bigcap_{j=1}^{\infty} E_j \) is measurable.

Definition 1. Given \( E \subseteq \mathbb{R} \) let \( m^*(E) = \inf_{E \subseteq \bigcup I_j} \sum_j |I_j| \) where the infimum is taken over coverings of \( E \) by collections \( \{I_j\} \) of open intervals and \( |I_j| \) is the usual length.

Lemma 1. If \( A \subseteq B \subseteq \mathbb{R} \), the \( m^*(A) \leq m^*(B) \).

Proof. This is clear since \( A \subseteq B \subseteq \bigcup I_j \) for each open covering of \( B \). \( \square \)

Lemma 2. For sets \( A, B \subseteq \mathbb{R} \), \( m^*(A \cup B) \leq m^*(A) + m^*(B) \).

Proof. This is clear. If \( A \subseteq \bigcup I_j \) and \( B \subseteq \bigcup J_i \), then \( A \cup B \subseteq \bigcup I_j \cup J_i \). \( \square \)

Definition 2. A set \( E \subseteq \mathbb{R} \) is said to be measurable if for all \( A \subseteq \mathbb{R} \), \( m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \).

Lemma 3. A set \( E \subseteq \mathbb{R} \) is measurable if and only if \( E^c \) is measurable.

Proof. This follows because the equation \( m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \) is symmetric in \( E \) and \( E^c \). \( \square \)

Lemma 4. To show that \( E \subseteq \mathbb{R} \) is measurable, it is enough to show that \( m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \) for an arbitrary \( A \subseteq \mathbb{R} \).

Proof. Since \( A = (A \cap E) \cup (A \cap E^c) \), it follows by lemma (2) that \( m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c) \) holds automatically. \( \square \)

Lemma 5. Suppose that \( E_1 \) and \( E_2 \) are measurable sets. Then \( E_1 \cap E_2 \) is measurable.

Proof. Let \( A \subseteq \mathbb{R} \), and note that \( A \cap (E_1 \cap E_2)^c = ((A \cap E_1) \cap E_2) \cup (A \cap E_2^c) \). Then
\[
m^*(A \cup (E_1 \cap E_2)) + m^*(A \cap (E_1 \cap E_2)^c) \leq m^*((A \cap E_1) \cap E_2) + m^*((A \cap E_1) \cap E_2^c) + m^*(A \cap E_2^c) = m^*(A \cap E_1) \leq m^*(A \cap E_2) = m^*(A) \text{.}
\]

The first inequality holds by lemma (2), the second holds because \( E_2 \) is measurable, and the last holds because \( E_1 \) is measurable. This is enough by lemma (4). \( \square \)

Lemma 6. Suppose that \( E_1 \) and \( E_2 \) are measurable sets. Then \( E_1 \cup E_2 \) is measurable.

Proof. This follows immediately from lemma (5) and lemma (3). \( \square \)

Lemma 7. Suppose that \( E_1, \ldots, E_n \) is a collection of measurable sets. Then \( \bigcap_{j=1}^{n} E_j \) is measurable.
Lemma 9. Suppose that $E_n = E_1 \cap \cdots \cap E_{n-1}$ is measurable. Since $E \cap E_n = E_1 \cap \cdots \cap E_n$, we are finished. \hfill \Box

Lemma 8. Suppose that $\{E_i\}$ is a collection of measurable sets. Then there are measurable sets $\{F_i\}$ such that

- $F_i \cup F_j = \mathbb{R}$ for $i \neq j$
- $\bigcap_{i=1}^{\infty} E_j = \bigcap_{i=1}^{\infty} F_j$

Proof. Let

\[
F_1 = E_1 \\
F_2 = E_2 \cup E_1^c \\
F_3 = E_3 \cup (E_2 \cap E_1)^c \\
\vdots \\
F_j = E_j \cup (E_{j-1} \cap \cdots \cap E_1)^c \\
\vdots
\]

By lemmas (3,6,7), $F_i$ is measurable for all $i$. If $j > i$, then

\[
F_j \cup F_i = \left( E_j \cup (E_{j-1} \cap \cdots \cap E_1)^c \right) \cup \left( E_i \cup (E_{i-1} \cap \cdots \cap E_1)^c \right) = E_j \cup E_{j-1}^c \cup \cdots \cup E_i^c \cup \cdots \cup E_1^c = \mathbb{R}
\]

since both $E_i$ and $E_i^c$ appear in the union. Furthermore, note that since $E_i \subseteq F_i$, it follows that $\cap E_i \subseteq \cap F_i$. If $x \in \cap F_i$, then by definition $x \in E_i \cup (E_{i-1} \cap \cdots \cap E_1)^c$. Of course, it is now enough to show that $x \in E_i$ for all $i$, that is, it is enough to show that the set $A = \{t \mid x \notin E_i\}$ is empty. Note that $1 \notin A$ by definition. If $A \neq \emptyset$, then there is a smallest element, $1 < a \in A$. Thus, $x \in E_{a-1} \cap \cdots \cap E_1$, but $x \notin E_a$. But $x \notin F_a = E_a \cup (E_{a-1} \cap \cdots \cap E_1)^c$, which must then imply that $x \in (E_{a-1} \cap \cdots \cap E_1)^c$, and this is a contradiction. \hfill \Box

Lemma 9. Suppose that $F_1, \ldots, F_n$ are measurable sets and $F_i \cup F_j = \mathbb{R}$ for $i \neq j$. Then $F_i^c \cap (F_1 \cap \cdots \cap F_{n-1})^c = \emptyset$. \hfill \Box

Proof. This is clear since $(F_i^c \cap (F_1 \cap \cdots \cap F_{n-1}))^c = (F_i \cup (F_1 \cap \cdots \cap F_{n-1})) = (F_i \cup F_1) \cap \cdots \cap (F_i \cup F_n) = \mathbb{R} \cap \cdots \cap \mathbb{R} = \mathbb{R}$. \hfill \Box

Lemma 10. Suppose that $E_1$ and $E_2$ are measurable sets and $E_1 \cap E_2 = \emptyset$. The $m^*(E_1) + m^*(E_2) = m^*(E_1 \cup E_2)$. \hfill \Box

Proof. Since $E_1$ is measurable, we have

\[
m^*(E_2 \cap E_1) + m^*(E_2 \cap E_1^c) = m^*(E_2).
\]

But $E_1 \cap E_2 = \emptyset$, and $E_2 \cap E_1^c = E_2$, so the result is proved. \hfill \Box

Lemma 11. Suppose that $F_1, \ldots, F_n$ are measurable sets $F_i \cup F_j = \mathbb{R}$ for $i \neq j$. Then $m^*(A \cap (F_1 \cap \cdots \cap F_n)^c) = \sum_{i=1}^{n} m^*(A \cap F_i^c)$. \hfill \Box

Proof. We do induction on $n$. If $n = 1$ this is a tautology. If $n > 1$, then write

\[
m^*(A \cap (F_1 \cap \cdots \cap F_n)^c) = m^*(A \cap ((F_1 \cap \cdots \cap F_{n-1})^c \cup F_n^c)) = m^*((A \cap F_n^c) \cup (A \cap ((F_1 \cap \cdots \cap F_{n-1})^c)) = m^*(A \cap F_n^c) + m^*(A \cap (F_1 \cap \cdots \cap F_{n-1})^c)
\]

where the third equality holds because $F_n^c$ and $(F_1 \cap \cdots \cap F_{n-1})^c$ are measurable and disjoint (see lemmas (9,10)), and the fourth inequality holds by the induction hypothesis. \hfill \Box

Lemma 12. Suppose that $\{E_j\}$ is a countable collection of measurable sets. Then $\sum_{i=1}^{\infty} m^*(A \cap E_j^c) \geq m^*(A \cap \bigcap_{i=1}^{\infty} E_j^c)$. \hfill \Box
Proof. Let $\epsilon > 0$ be given and suppose that $m^*(A \cap E_j^c) = e_j$. Then (by the definition of outer measure) there is $\{I_i^{(j)}\}$ such that $\cup_i I_i^{(j)} \supseteq (A \cap E_j^c)$ and $\sum |I_i^{(j)}| < e_j + \epsilon/2^j$. Of course

$$\cup_j \cup_i I_i^{(j)} \supseteq \cup_j (A \cap E_j^c) = A \cap (\cup_j E_j)^c,$$

so clearly $m^*(A \cap (\cup_j E_j)^c) \leq \sum_j \sum_i |I_i^{(j)}| = \sum_j e_j + \epsilon/2^j = \epsilon + \sum_j e_j = \epsilon + \sum_j m^*(A \cap E_j^c)$. Since this is true for each $\epsilon > 0$, this completes the proof. \[\square\]

**Lemma 13.** Suppose that $\{E_j\}$ is a countable collection of measurable sets. Then $\cap_j E_j$ is measurable.

**Proof.** First, we find $\{F_j\}$ such that $F_i \cup F_j = \mathbb{R}$ for $i \neq j$ and $\cap_{i=1}^\infty E_j = \cap_{i=1}^\infty F_j$ (as guaranteed to exist by lemma (8)). Let $G = \cap_{i=1}^\infty F_i = \cap_{i=1}^\infty E_i$ and $G_n = F_1 \cap \cdots \cap F_n$. Then $G_n$ is measurable for all $n \in \mathbb{N}$ by lemma (7), and hence

$$m^*(A) = m^*(A \cap G_n) + m^*(A \cap G_n^c)$$

for all $A \subseteq \mathbb{R}$. Since $G_n = F_1 \cap \cdots \cap F_n \subseteq \cap_{i=1}^\infty F_i = G$, it follows that $m^*(A \cap G_n) \geq m^*(A \cap G)$ (by lemma (1)), and thus

$$m^*(A) = m^*(A \cap G_n) + m^*(A \cap G_n^c) \geq m^*(A \cap G) + m^*(A \cap G_n^c).$$

Now by lemma (11), we can write

$$m^*(A) \geq m^*(A \cap G) + m^*(A \cap G_n^c) = m^*(A \cap G) + \sum_{i=1}^n m^*(A \cap F_i^c)$$

and this holds for all $n$. We conclude that

$$m^*(A) \geq m^*(A \cap G) + \sum_{i=1}^\infty m^*(A \cap F_i^c).$$

But by lemma (12),

$$\sum_{i=1}^\infty m^*(A \cap F_i^c) \geq m^*(A \cap (\cup_{i=1}^\infty F_i^c)) = m^*(A \cap (\cap_{i=1}^\infty F_i)^c),$$

and this certainly completes the proof, since now we have that

$$m^*(A) \geq m^*(A \cap G) + \sum_{i=1}^\infty m^*(A \cap F_i^c) \geq m^*(A \cap G) + m^*(A \cap (\cap_{i=1}^\infty F_i)^c) = m^*(A \cap G) + m^*(A \cap G^c),$$

which is enough by lemma (4). \[\square\]

**Problem 3.** Let $\{f_j\}$ be a sequence of continuous, real valued functions on $\mathbb{R}$ with the property that $\{f_j(x)\}$ is unbounded for all $x \in \mathbb{Q}$. Use the Baire Category theorem to prove that it cannot then be true that whenever $t$ is irrational then the sequence $\{f_j(t)\}$ is bounded. Note: this is a somewhat challenging problem. You need to start by deciding what complete metric space the Baire Category theorem should give you information about.

**Solution.** We will show that if the sequence $\{f_j(t)\}$ is bounded at each irrational $t$, then we can write $\mathbb{R}_{\geq 0}$ as a countable union of nowhere dense sets, which contradicts the Baire Category theorem (since $\mathbb{R}_{\geq 0}$ is a complete metric space).

For $j \in \mathbb{N}$, let

$$A_j = \left\{ t \in \mathbb{R} - \mathbb{Q} \mid \{|f_j(t)|\} \text{ is bounded (not strictly) by } j \right\}.$$

Let $\{q_j\}$ be an enumeration of $\mathbb{Q}$, and let $Q_j = \{q_j\}$. By hypothesis, every $t \in \mathbb{R} - \mathbb{Q}$ is in some $A_j$, and of course $\cup_{j=1}^\infty Q_j = \mathbb{Q}$, so clearly

$$\cup_{j=1}^\infty Q_j \cup (\cup_{j=0}^\infty nfty A_j) = \mathbb{R}_{\geq 0}.$$

Also, this is a countable union, so to obtain the contradiction it is enough to show that $Q_j$ and $A_j$ for all $j \in \mathbb{N}$ are nowhere dense.

That $Q_j$ is nowhere dense is immediate, since $Q_j = \{q_j\}$ consists of a single point.

For $A_j$, we first show that it is closed. Suppose that $r$ is an accumulation point of $A_j$. We need to demonstrate that $|f_N(r)| < j$ for all $N \in \mathbb{N}$, that is, we need to show that $r \in A_j$. So fix $N \in \mathbb{N}$. Then given $\epsilon > 0$, there is $\delta > 0$ (since $f_N$ is
continuous) such that \(|x - r| < \delta\) implies that \(|f_N(x) - f_N(r)| < \epsilon\). But \(r\) is an accumulation point of \(A_j\), so there is \(t \in A_j\) such that \(|t - r| < \delta\) and hence \(|f_N(r)| < |f_N(t)| + \epsilon \leq j + \epsilon\). Since \(\epsilon\) was arbitrary, we conclude that \(f_N(r) \leq j\). But \(N\) was also arbitrary. Hence the conclusion that \(\{f_j(r)\}\) is bounded above by \(j\) and thus \(r \in A_j\) as required.

Next we show that \(A_j\) contains no open balls (this is enough since \(A_j\) is its own closure). In fact, this is easy because (by definition) \(\mathbb{Q} \cap A_j = \emptyset\), but there are rationals arbitrarily close to any irrational. To be precise, suppose that \(t \in A_j\) and \(\epsilon > 0\) is given. Then the ball \(B(t, \epsilon) = \{x \in \mathbb{R} \mid |x - t| < \epsilon\} \subset A_j\) because (by the density of the rationals in the reals) there is \(q \in \mathbb{Q}\) such that \(|q - t| < \epsilon\), but \(q \not\in A_j\).

We conclude that \(A_j\) is nowhere dense. By the Baire Category theorem, this completes the proof. \(\square\)