Please do all your work in this booklet and show all the steps.
Calculators and note-cards are not allowed.

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Problem 1. (10 pts.) Suppose that a particle moves according to a law of motion \( s(t) = -t^3 + 9t^2 - 24t - 3 \) for \( 0 \leq t \leq 10 \) where \( t \) is measured in seconds and \( s \) is measured in meters.

(a – 5 pts) What is the velocity of the particle after 3 seconds?

If \( s(t) \) gives the position of the particle at time \( t \), then the rate of change of distance with respect to time, the derivative of \( s(t) \), gives velocity. So we compute that \( v(t) = s'(t) = -3t^2 + 18t - 24 \). The velocity at time \( t = 3 \) is then \( v(3) = -3(3)^2 + 18(3) - 24 = 3 \text{ m/s} \).

(b – 5 pts) At what time does the particle begin to decelerate? (You are accelerating if acceleration is positive).

Acceleration is the rate of change of velocity with respect to time, or \( a(t) = v'(t) \). So we compute that \( a(t) = -6t + 18 \). Now \( a(t) = 0 \) if and only if \( 0 = -6t + 18 \), that is, if \( t = 3 \). Furthermore, \( a(0) = 18 > 0 \), while \( a(4) = -6 < 0 \). Because the sign of \( a(t) \) can only change at \( t = 3 \), we can conclude that \( a(t) \) is positive for \( -\infty < t < 3 \) and \( a(t) \) is negative for \( 3 < t < \infty \). Thus it is at \( t = 3 \) that we begin to decelerate.
Problem 2. (10 pts.) Let \( f(x) = \sqrt[3]{x} \).

(a – 5 pts) Compute the linearization of \( L(x) \) of the function \( f(x) \) at the point \( a = 8 \).

The equation for the linearization of a function \( f(x) \) at \( x = a \) is \( L(x) = f(a) + f'(a)(x - a) \). We see that \( f(8) = \sqrt[3]{8} = 2 \), \( f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \), and \( f'(8) = \frac{1}{3}8^{-\frac{2}{3}} = \frac{1}{3 \cdot 8^{\frac{2}{3}}} = \frac{1}{12} \). So the linearization at \( x = 8 \) is \( L(x) = 2 + \frac{1}{12}(x - 8) \).

(b – 5 pts) Use your answer above to estimate \( f(8.1) \).

We know that \( f(8.1) \approx L(8.1) = 2 + \frac{1}{12}(8.1 - 8) = 2 + \frac{1}{12} \frac{1}{10} = 2 + \frac{1}{120} = \frac{240}{120} + \frac{1}{120} = \frac{241}{120} \).
Problem 3. (18 pts.) Sketch the function \( f(x) = x^3 - 6x^2 + 9x + 3 \). Be sure to identify all local maximum, local minimum, and inflection points. Also, indicate the intervals for which the function is increasing, decreasing, concave up, and concave down.

(1) The first step is to compute the first and second derivatives:
\[
f'(x) = 3x^2 - 12x + 9, \quad \text{and} \quad f''(x) = 6x - 12
\]
(2) Next we set \( f'(x) = 0 \) and solve for \( x \):
\[
0 = 3x^2 - 12x + 9 \Rightarrow 0 = (3x - 3)(x - 3),
\]
so we conclude that \( f'(x) = 0 \) if \( x = 3 \) or 1.
(3) Now compute \( f''(3) \) and \( f''(1) \) to classify these points as local maxs, local mins, or otherwise.
\[
f''(3) = 6 > 0, \quad \text{so } x = 3 \text{ is a local min;}
\]
\[
f''(1) = -6 < 0 \quad \text{so } x = 1 \text{ is a local max.}
\]
(4) We can determine the intervals of increase and decrease from this information:
Because \( f'(x) = 0 \) only at \( x = 3, 1 \), it is enough to check the value of the derivative at one point in each of the intervals \( (-\infty, 1) \), \( (1, 3) \), and \( (3, \infty) \) to determine whether the graph is increasing or decreasing there. We see that
\[
f'(0) = 9 > 0, \quad \text{so the graph is increasing on } (-\infty, 1),
\]
\[
f'(2) = -3 < 0, \quad \text{so the graph is decreasing on } (1, 3),
\]
\[
f'(4) = 3(16) - 48 + 9 = 9 > 0 \quad \text{so the graph is increasing on } (3, \infty).
\]
(5) The next step is to set \( f''(x) = 0 \) and solve for \( x \):
\[
0 = 6x - 12 \quad \Rightarrow x = 2, \quad \text{so there is a possible inflection point at } x = 2.
\]
(6) At this point, we determine whether or not \( x = 2 \) is an inflection point:
\[
f''(1) = -6 < 0 \quad \text{and} \quad f''(4) = 12 > 0, \quad \text{so at } x = 2 \text{ the graph changes from concave down to concave up and } x = 2 \text{ is an inflection point.}
\]
(7) It remains to compute \( f(1), f(2) \) and \( f(3) \)
\[
f(1) = (1)^3 - 6(1)^2 + 9(1) + 3 = 7.
\]
\[
f(2) = (8) - 6(4) + 9(2) + 3 = 5.
\]
\[
f(3) = (27) - 6(9) + 9(3) + 3 = 3.
\]
(8) So a sketch of the graph is:

The function is increasing on \( (-\infty, 1) \) and \( (3, \infty) \), decreasing on \( (1, 3) \), concave down on \( (-\infty, 2) \), concave up on \( (2, \infty) \), has a local max at \( (1, 7) \), a local min at \( (3, 3) \), and an inflection point at \( (2, 5) \).
Problem 4. (10 pts.) Do either (a) or (b). Indicate clearly which you want graded.

(a – 15 pts) A rectangular bin with a square base and open top is to have a volume of 32 cubic meters and be made of sheet metal. What are the dimensions which minimize the number of square meters of sheet metal necessary to build the bin?

We are asked to: Minimize surface area given volume. Let \( x \) be the side length of the base, and let \( y \) be the height of the bin. Then our constraint equation is \( x^2 y = 32 \), because our bin is to have volume of 32 cubic meters. The surface area of the bin, \( S \) is \( S = x^2 + 4xy \) (the surface area of the base plus four times the surface area of one of the sides), and this gives us our objective equation.

From the constraint we know that \( y = \frac{32}{x^2} \), so

\[
S = x^2 + 4x \frac{32}{x^2} = x^2 + \frac{128}{x}.
\]

The derivative of \( S \) is \( S' = 2x - \frac{128}{x^2} \) and setting \( S' \) equal to zero we obtain

\[
0 = 2x - \frac{128}{x^2} \Rightarrow \frac{128}{x^2} = 2x \Rightarrow x^3 = 64 \Rightarrow x = \sqrt[3]{64} = 4.
\]

Now \( S'' = 2 + \frac{256}{x^3} \) is certainly positive when \( x = 4 \), so the critical point we have found is a minimum. From the constraint equation we can compute that when \( x = 4 \), the value of \( y \) is \( y = \frac{32}{16} = 2 \). Thus the dimensions that minimize surface area are side length \( x = 4 \) meters and height \( y = 2 \) meters.

(b – 15 pts) A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm³/s, how fast is the water level rising when the water level is 5 cm deep?

Let \( V \) be the volume of water in the cone, \( h \) be the height of the water, and \( r \) be the radius of the cup at the water level \( h \). We want to compute to compute \( \frac{dh}{dt} \), given that \( \frac{dV}{dt} = 3 \text{ cm}^3/\text{s} \). Note that the volume of a cone of height \( h \) and radius \( r \) is \( V = \frac{1}{3} \pi r^2 h \). The first step is to remove the variable \( r \) from the equation, which can be done using similar triangles—in particular, \( \frac{10}{3} = \frac{h}{r} \) (the ratio of the height of the cup to its radius is equal to the ratio of the height of the water to the radius of the cup at the water height) and thus \( r = \frac{3h}{10} \). So

\[
V = \frac{1}{3} \pi \left( \frac{3h}{10} \right)^2 h = \frac{3\pi}{100} h^3.
\]

Now we use implicit differentiation to compute the derivative of both side of this equation with respect to \( t \), or we note that the chain rule implies that \( \frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} \), so that \( \frac{dh}{dt} = \frac{dV}{dh} \frac{dV}{dt} \). The derivative of \( V \) with respect to \( h \) is \( \frac{dV}{dh} = \frac{9\pi}{100} h^2 \), and at \( h = 5 \text{ cm} \), \( \frac{9\pi}{100} 5^2 = \frac{9\pi}{4} \). Thus when \( h = 5 \), the derivative of \( h \) with respect to \( t \) is

\[
\frac{dh}{dt} = \frac{32}{9\pi} \text{ cm/s}.
\]
Problem 5. (12 pts.) Consider the following graph, then match each of the entries in the left column with one entry from the right column.

| H | M such that if \( x > M \) then \( |f(x) - 1| < 1 \) |
|---|---|
| D | Slope of the tangent at the point \( c \) satisfying the conclusion of the Mean Value Theorem for the interval \([2, 7]\) |
| J | Slope of the tangent at \( x = 7 \) |
| G | \( \lim_{x \to -\infty} |f(x) - x| \) |
| A | \( f(7) \) |
| B | interval such that \( f''(x) < 0 \) |
| F | interval such that \( f''(x) > 0 \) |
| I | \( x \) such that \( f''(x) = 0 \) |
| E | \( \epsilon > 0 \) such that if \( x > 9 \), then \( |f(x) - 1| \leq \epsilon \) |
| L | the slope of the linearization to \( f \) at \( a = 7.5 \) |
| C | \( \lim_{x \to -\infty} f(x) \) |
| K | \( M > 0 \) such that if \( |x - 9| < \frac{1}{2} \), then \( |f(x)| > M \) |

(a) 5  (b) (3, 7)  (c) 1  (d) 2/5  (e) Does not exist  (f) (8, 9)  (g) 0  (h) 10  (i) 7.5  (j) -1  (k) 2  (l) -2