Solutions to Assignment 7

Math 217, Fall 2002

4.3.10 Find a basis for the null space of the following matrix: \( A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \).

We need to find a basis for the solutions to the equation \( Ax = 0 \). To do this we first put \( A \) in row reduced echelon form. The result (according to the computer) is:
\[
\begin{bmatrix}
1 & 0 & -5 & 0 & 7 \\
0 & 1 & -4 & 0 & 6 \\
0 & 0 & 0 & 1 & -3
\end{bmatrix}
\]

From this we can read the general solution, 
\[
x = \begin{bmatrix} 5x_3 - 7x_5 \\ 4x_3 - 6x_5 \\ x_5 \\ 3x_5 \\ x_5 \end{bmatrix}
\]
write this as 
\[
x = x_3 \begin{bmatrix} 5 \\ 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 1 \\ 1 \end{bmatrix}
\]

Because these two vectors are clearly not multiples of one another, they also give a basis. So a basis for null(\( A \)) is
\[
\left\{ \begin{bmatrix} 5 \\ 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} , \begin{bmatrix} -7 \\ -6 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}
\]

3.3.28 Let \( \mathbf{v}_1 = \begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix} \), \( \mathbf{v}_2 = \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix} \), \( \mathbf{v}_3 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix} \). It can be verified that \( \mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0} \). Use this information to find a basis for \( H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \).

By the Spanning Set Theorem, some subset of the \( \mathbf{v}_i \) is a basis for \( H \). In class we showed how to find this subset. We simply remove any of the vectors involved in a non-trivial linear relation. So I choose to remove \( \mathbf{v}_3 \) (I could have
removed any of the \( \mathbf{v}_i \) because they each occur with a non-zero coefficient in the dependency relation \( \mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0} \). The remaining vectors then give a basis for \( H \). We know they span by the Spanning Set Theorem. They are also linearly independent, because they are not multiples of one another.

**4.3.30** Let \( S = \{ \mathbf{v}_1, \ldots, \mathbf{v}_k \} \) be a set of \( k \) vectors in \( \mathbb{R}^n \), with \( k > n \). Use a theorem from Chapter 1 to explain why \( S \) cannot be a basis for \( \mathbb{R}^n \).

Theorem 8, page 69, claims that any set \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) in \( \mathbb{R}^n \) is linearly dependent if \( p > n \). That is exactly the situation we find ourselves in (expect that we have used the letter \( k \) instead of \( p \)). Thus \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_k \} \) is linearly dependent, and cannot be a basis.

**4.3.32** Suppose that \( T \) is a one-to-one transformation. Show that if the set of images \( \{ T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p) \} \) is linearly dependent, then \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) is linearly dependent.

**This problem should say that \( T \) is a linear transformation (the book has a typo).**

If the set \( \{ T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p) \} \) is linearly dependent, then there are \( c_1, \ldots, c_p \in \mathbb{R} \) not all zero such that \( c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = \mathbf{0} \). Because \( T \) is a linear transformation, we can rewrite this as \( T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = \mathbf{0} \). We know that \( T(\mathbf{0}) = \mathbf{0} \), so because \( T \) is one-to-one it must be the case that \( c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \). This gives a dependency relation on the \( \mathbf{v}_i \) (recall that not all the \( c_i \) are zero), and thus \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) is linearly dependent.

**4.4.14** The set \( \mathcal{B} = \{ 1 - t^2, t - t^2, 2 - 2t + t^2 \} \) is a basis for \( \mathbb{P}_2 \). Find the coordinate vector of \( \mathbf{p}(t) = 3 + t - 6t^2 \) relative to \( \mathcal{B} \).

We need to write \( \mathbf{p} \) in terms of the basis \( \mathcal{B} \), that is, find \( x_1, x_2, x_3 \in \mathbb{R} \) such that
\[
x_1(1 - t^2) + x_2(t - t^2) + x_3(2 - 2t + t^2) = 3 + t - 6t^2.
\]
Multiplying things out, we get
\[
(x_1 + 2x_3) + (x_2 - 2x_3)t + (-x_1 - x_2 + x_3)t^2 = 3 + t - 6t^2.
\]
Thus we have to solve the three linear equations:
\[
\begin{align*}
x_1 + 2x_3 &= 3 \\
x_2 - 2x_3 &= 1 \\
-x_1 - x_2 + x_3 &= -6
\end{align*}
\]

We form the augmented matrix for this system,
\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & -2 & 1 \\
-1 & -1 & 1 & -6
\end{bmatrix}.
\]
In row reduced echelon form, this is the matrix \[
\begin{bmatrix}
1 & 0 & 0 & 7 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & -2
\end{bmatrix}
\]. So we see that

\( x_1 = 7, \ x_2 = -3, \) and \( x_3 = -2. \)

This implies that \( 7(1 - t^2) + (-3)(t - t^2) + (-2)(2 - 2t + t^2) = 3 + t - 6t^2, \) so

\[
[3 + t - 6t^2]_B = \begin{bmatrix}
7 \\
-3 \\
-2
\end{bmatrix} \in \mathbb{R}^3.
\]

**4.4.20** Suppose that \( \{\mathbf{v}_1, \ldots, \mathbf{v}_4\} \) is a linearly dependent spanning set for a vector space \( V. \) Show that each \( \mathbf{w} \) in \( V \) can be expressed in more than one way as a linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_4. \)

Because the \( \mathbf{v}_i \) are linearly dependent, there are \( c_1, \ldots, c_4 \in \mathbb{R} \) not all zero such that \( c_1 \mathbf{v}_1 + \cdots + c_4 \mathbf{v}_4 = \mathbf{0}. \) Given \( \mathbf{w} \) there are \( d_1, \ldots, d_4 \in \mathbb{R} \) such that \( d_1 \mathbf{v}_1 + \cdots + d_4 \mathbf{v}_4 = \mathbf{w} \) (because the \( \mathbf{v}_i \) are a spanning set). Consider the linear combination \( (c_1 + d_1) \mathbf{v}_1 + \cdots + (c_4 + d_4) \mathbf{v}_4. \) We have that \( (c_1 + d_1) \mathbf{v}_1 + \cdots + (c_4 + d_4) \mathbf{v}_4 = (c_1 \mathbf{v}_1 + \cdots + c_4 \mathbf{v}_4) + (d_1 \mathbf{v}_1 + \cdots + d_4 \mathbf{v}_4) = \mathbf{0} + \mathbf{w} = \mathbf{w}. \) This constitutes a different linear combination than \( d_1 \mathbf{v}_1 + \cdots + d_4 \mathbf{v}_4 \) because not all of the \( c_i \) are zero, and hence for some \( i \) between 1 and 4, we have that \( c_i + d_i \neq d_i. \)

**4.4.32** Let \( \mathbf{p}_1(t) = 1 + t^2, \ \mathbf{p}_2(t) = 2 - t + 3t^2, \ \mathbf{p}_3(t) = 1 + 2t - 4t^2. \)

(a) Use coordinate vectors to show that these polynomials form a basis for \( \mathbb{P}_2. \)

We know that a basis for \( \mathbb{P}_2 \) is \( \mathcal{B} = \{1, t, t^2\} \). It is not difficult to see that if the \( \mathbf{p}_i \) are dependent, then so are the images \( [\mathbf{p}_1]_\mathcal{B}, [\mathbf{p}_2]_\mathcal{B}, [\mathbf{p}_3]_\mathcal{B}. \) That is, if \( \mathbf{0} = c_1 \mathbf{p}_2 + c_2 \mathbf{p}_2 + c_3 \mathbf{p}_3 \) is a nontrivial dependency (i.e., not all \( c_i \) are zero), then taking the coordinate mapping of both sides of the equation yields the dependency relation \( \mathbf{0} = [\mathbf{0}]_\mathcal{B} = [c_1 \mathbf{p}_2 + c_2 \mathbf{p}_2 + c_3 \mathbf{p}_3]_\mathcal{B} = c_1 [\mathbf{p}_2]_\mathcal{B} + c_2 [\mathbf{p}_2]_\mathcal{B} + c_3 [\mathbf{p}_3]_\mathcal{B}. \)

We also know that if \( [\mathbf{p}_1]_\mathcal{B}, [\mathbf{p}_2]_\mathcal{B}, [\mathbf{p}_3]_\mathcal{B} \) spans \( \mathbb{R}^3, \) then the \( \mathbf{p}_i \) span \( \mathbb{P}_2. \) That is, assuming the \( [\mathbf{p}_i]_\mathcal{B} \) span, we know that for each \( f \in \mathbb{P}_2 \) there are \( d_i \) such that \( [f]_\mathcal{B} = d_1 [\mathbf{p}_1]_\mathcal{B} + d_2 [\mathbf{p}_2]_\mathcal{B} + d_3 [\mathbf{p}_3]_\mathcal{B} = [d_1 \mathbf{p}_1 + d_2 \mathbf{p}_2 + d_3 \mathbf{p}_3]_\mathcal{B}, \) and because the coordinate mapping is one-to-one, this implies that \( f = d_1 \mathbf{p}_1 + d_2 \mathbf{p}_2 + d_3 \mathbf{p}_3. \)

So to show the \( \mathbf{p}_i \) are a basis of \( \mathbb{P}_2, \) it is enough to show that the \( [\mathbf{p}_i]_\mathcal{B} \) are a basis of \( \mathbb{R}^3. \) By the Invertible Matrix Theorem, this set will be a basis if and only if the matrix \( \begin{bmatrix} [\mathbf{p}_1]_\mathcal{B} & [\mathbf{p}_2]_\mathcal{B} & [\mathbf{p}_3]_\mathcal{B} \end{bmatrix} = \begin{bmatrix}
1 & 2 & 1 \\
0 & -1 & 2 \\
1 & 3 & -4
\end{bmatrix} \) has a pivot in every row (because then by IMT the columns will span and will be linearly independent). So we put

\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & -1 & 2 \\
1 & 3 & -4
\end{bmatrix}
\] in row reduced echelon form.
form and obtain: \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]. We conclude that the set \( \{p_1, p_2, p_3\} \) does form a basis of \( \mathbb{P}_2 \).

(b) Consider the basis \( \mathcal{B} = \{p_1, p_2, p_3\} \) for \( \mathbb{P}_2 \). Find \( q \) in \( \mathbb{P}_2 \) given that \( [q]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \).

Because \( [q]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \), we know that

\[
q = -3p_1 + p_2 + 2p_3 = -3(1+t^2) + 2(2-t+3t^2) + 2(1+2t-4t^2) = 1 + 3t - 8t^2.
\]

4.5.12 Find the dimension of the vector space spanned by \[
\begin{bmatrix} 1 \\ -2 \\ 0 \\
-3 \\
0 \\
7
\end{bmatrix}, \quad
\begin{bmatrix} -3 \\ 4 \\ 1 \\
0 \\
1 \\
5
\end{bmatrix}, \quad
\begin{bmatrix} -8 \\ 6 \\ 5 \\
0 \\
0 \\
1
\end{bmatrix}, \quad
\begin{bmatrix} -3 \\ 0 \\ 7 \\
0 \\
1 \\
5
\end{bmatrix}.
\]

To find the dimension, we need to find the number of elements in a basis. So we form the matrix \[
\begin{bmatrix}
1 & -3 & -8 & -3 \\
-2 & 4 & 6 & 0 \\
0 & 1 & 5 & 7
\end{bmatrix}
\] and count the number of pivot columns (this because we know that \( \text{dim}(\text{Col}(A)) \) for a matrix \( A \) is precisely the number of pivot columns of \( A \)). So we put \[
\begin{bmatrix}
1 & -3 & -8 & -3 \\
-2 & 4 & 6 & 0 \\
0 & 1 & 5 & 7
\end{bmatrix}
\] in row reduced echelon form, obtaining the matrix: \[
\begin{bmatrix}
1 & 0 & 7 & 0 \\
0 & 1 & 5 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]. This has 3 pivot columns, so the dimension of the subspace spanned by the vectors given is 3.

4.5.22 The first four Laguerre polynomials are \( 1 \), \( 1-t \), \( 2-4t+t^2 \), and \( 6-18t+9t^3-t^3 \). Show that these polynomials form a basis of \( \mathbb{P}_3 \).

Consider the basis \( \mathcal{B} = \{1, t, t^2, t^3\} \) of \( \mathbb{P}_3 \). Utilizing the same arguments as we did in question 4.4.32, we know that it is enough to show that the images of these polynomials form a basis of \( \mathbb{R}^4 \) under to coordinate mapping. Their images under this mapping are: \( [1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [1-t]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [2-4t+t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \quad [6-18t+9t^3-t^3]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -18 \\ 0 \\ 9 \\ -1 \end{bmatrix} \).
and \([6 - 18t + 9t^3 - t^3]_B = \begin{bmatrix} 6 \\ -18 \\ 9 \\ -1 \end{bmatrix}.

By IMT, these will form a basis if and only if the matrix

\[
\begin{bmatrix}
1 & 1 & 2 & 6 \\
0 & -1 & -4 & -18 \\
0 & 0 & 1 & 9 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

is invertible. This matrix will be invertible if and only if its determinate is non-zero (theorem 4 page 194). The computer tells me that the determinate of this matrix is equal to 1. So we conclude that the given polynomials do form a basis.

4.5.32 Let \(H\) be a nonzero subspace of \(V\), and suppose \(T\) is a one-to-one (linear) mapping of \(V\) into \(W\). Prove that \(\dim T(H) = \dim H\). If \(T\) happens to be a one-to-one mapping of \(V\) onto \(W\), then \(\dim V = \dim W\). Isomorphic finite dimensional vector spaces have the same dimension.

We know that if \(v_1, \ldots, v_p\) is linearly independent, then because \(T\) is one-to-one, \(T(v_1), \ldots, T(v_p)\) is linearly independent (that was an earlier homework problem, 4.3.32). We showed in class that if \(v_1, \ldots, v_p\) spans \(H\), then \(T(v_1), \ldots, T(v_p)\) spans \(T(H)\). This means that because the \(v_1, \ldots, v_p\) form a basis of \(H\), the \(T(v_1), \ldots, T(v_p)\) give a basis of \(T(H)\). They both have the same size, so both spaces have the same dimension.

When \(T\) is also onto, then taking \(H = V\) we have that \(T(V) = W\), and the statement follows from what we have already done.