Math 217, Linear Algebra, Fall 2002
Exam 2 Solutions

1. (8pts) Let $A$ be the matrix

$$A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\
31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\
41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 \\
51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 & 59 & 60 \\
61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 & 69 & 70 \\
71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 \\
81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 & 89 & 90 \\
91 & 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 & 100 \\
\end{bmatrix}.$$ 

It turns out that $Ax = 0$ is true for both

$$x = \begin{bmatrix}
1 \\
-2 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix}
65 \\
-80 \\
0 \\
0 \\
5 \\
4 \\
3 \\
2 \\
1 \\
0 \\
\end{bmatrix}.$$ 

Does $Ax = b$ have a solution for all possible $b \in \mathbb{R}^{9}$? Explain. (Hint: don’t use your calculator).

We know that $\text{rank}(A) + \text{dim}(\text{Nul}(A)) = 10$, the number of columns. Each of the given vectors is an element in $\text{Nul}(A) = \{x \in \mathbb{R}^{10} \mid Ax = 0\}$, and they are clearly independent (because there are only two, we may check to see that they are not multiples of each other, which is true), so $\text{dim}(\text{Nul}(A)) \geq 2$. This means that $\text{rank}(A) \leq 8$, and in particular that $A$ can have at most 8 pivots. This means that in row reduced echelon form, $A$ has a row of zeros, and thus the equation $Ax = b$ will not be consistent for all $b \in \mathbb{R}^{9}$.

2. (10pts) Suppose that the matrix $B = \begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 3 & 2 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}$ was obtained from $A$ by the following row and column operations (performed in the order listed).

1. I replaced row 4 with row 4 plus 2 times row 3.
2. I replaced col 1 with col 1 minus 3 times col 4.
3. I swapped rows 2 and 4.
4. I multiplied row 3 row by the scalar 3.
5. I replaced row 4 with row 4 plus 4 times row 3.
6. Finally I swapped columns 3 and 1.

What is det A? Show your work.

We know that Row and Col replacements do not change the determinate, if A and B are equivalent via a row or column swap then \(-\det A = \det B\) and finally, if we obtained B from A by multiplying a row of A by the scalar c, then \(c\det A = \det B\).

To obtain B from A, I made two Row/Col swaps, and I multiplied a row by 3. So, following the rules above \((-1)(3)(-1)\) \(\det A = \det B\).

In more detail, if \(A_i\) is the matrix obtained after making the \(i\)th calculation (so that \(A_0 = A\) and \(A_6 = B\)), then

\[
\begin{align*}
\det A_1 &= \det A_0 \\
\det A_2 &= \det A_1 \\
\det A_3 &= -\det A_2 \\
\det A_4 &= 3\det A_3 \\
\det A_5 &= \det A_4 \\
\det A_6 &= -\det A_5
\end{align*}
\]

Putting this all together, \(\det B = -\det A_5 = -\det A_4 = -3\det A_3 = 3\det A_2 = 3\det A_1 = 3\det A\).

Now because B is a diagonal matrix, its determinate is just the product of the diagonal entries (this is a theorem), so \(\det B = 9\).

So \(9 = 3\det A\), and we conclude that \(\det A = 3\).

3.(6pts) Each of the sets

\[
\begin{align*}
B_1 &= \{1 - t^2, 1 + t, 1 - t, 1 + 2t + t^2, 4 + 4t + 4t^2\}, \\
B_2 &= \{2 + t + 2t^2, 2 + 2t, 1 - 2t - 3t^2\}, \text{ and} \\
B_3 &= \{1 + t + t^2, 3 + t - 8t^2, 5t - 2t^2, 4 + 7t - 9t^2\}
\end{align*}
\]

span \(P_2\). One of them is a basis. Which one is it? Explain.

We know from the spanning theorem that any spanning set contains a basis. This means that each of \(B_1, B_2,\) and \(B_3\) contains a basis. If a set other than \(B_2\) were a basis, we would get a contradiction, because another smaller set would contain a basis of a smaller size (we know that all bases of a vector space \(V\) have the same size). So the answer must be \(B_2\).

4.(16pts) It can be shown that

\[
A = \begin{bmatrix}
-2 & 4 & -2 & -4 \\
2 & -6 & -3 & 1 \\
-3 & 8 & 2 & -3
\end{bmatrix}
\]
is equivalent to

\[ B = \begin{bmatrix}
1 & 0 & 6 & 5 \\
0 & 2 & 5 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}. \]

Write down a basis for each of the following (be sure to explain or show your work):

(a) \( \text{Col}(A) \): A basis of \( \text{Col}(A) \) is given by the pivot columns of \( A \) (this was a theorem we proved in class). These are the same as the pivot columns of \( B \), so a basis for \( \text{Col}(A) \) is \[
\left\{ \begin{bmatrix}
-2 \\
2 \\
-3
\end{bmatrix},
\begin{bmatrix}
4 \\
-6 \\
8
\end{bmatrix} \right\}.
\]

(b) \( \text{Row}(A) \): We know that a basis for \( \text{Row}(A) \) is given by the non-zero rows of \( A \) in row echelon form. We can just read this off of \( B \), \[
\left\{ \begin{bmatrix}
1 & 0 & 6 & 5
\end{bmatrix},
\begin{bmatrix}
0 & 2 & 5 & 3
\end{bmatrix} \right\}.
\]

(c) \( \text{Nul}(A) \): To compute \( \text{Nul}(A) \) we are looking for the solutions to the equations \( Ax = 0 \). So we augment \( A \) with \( 0 \) and put the result in row reduced echelon form. This is the same, of course, as augmenting \( B \) with \( 0 \) and finding the row reduced echelon form. The result is \[
\begin{bmatrix}
1 & 0 & 6 & 5 \\
0 & 1 & 5/2 & 3/2 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
and the general solution to this set of equations is \( x_3 \begin{bmatrix}
-6 \\
-5/2 \\
1
\end{bmatrix} + x_4 \begin{bmatrix}
-5 \\
-3/2 \\
0
\end{bmatrix} \)

or \( \text{span} \left( \begin{bmatrix}
-6 \\
-5/2 \\
1
\end{bmatrix},
\begin{bmatrix}
-5 \\
-3/2 \\
0
\end{bmatrix} \right) \). Thus \( \left\{ \begin{bmatrix}
-6 \\
-5/2 \\
1
\end{bmatrix},
\begin{bmatrix}
-5 \\
-3/2 \\
0
\end{bmatrix} \right\} \) is a basis for \( \text{Nul}(A) \) (the two vectors given are not multiples of each other, so they are linearly independent—note that this only works if we are dealing with only two vectors).

5.(10pts) Let \( S \) be the parallelogram determined by the vectors \( b_1 = \begin{bmatrix}
-2 \\
3
\end{bmatrix} \) and \( b_2 = \begin{bmatrix}
-2 \\
5
\end{bmatrix} \), and let \( A = \begin{bmatrix}
6 & -2 \\
-3 & 2
\end{bmatrix} \). Compute the area of the image of \( S \) under the mapping \( x \rightarrow Ax \). Show your work.

The formula the book gives us for this situation is that if we let \( T(x) \) be the linear transformation \( T(x) = Ax \), then \( \{\text{area of } T(S)\} = \det A \cdot \{\text{area of } S\} \). We also know that the area of the parallelogram \( S \) is the absolute value of the determinate of the matrix whose columns are the vectors we used to define \( S \) in the first place.

Thus \( \{\text{area of } S\} = |\det \begin{bmatrix}
-2 & -2 \\
3 & 5
\end{bmatrix}| = | -10 + 6 | = 4 \), \( \det A = 12 - 6 = 6 \), so \( \{\text{area of } T(S)\} = 24 \) square units.
Determine if the set of all matrices of the form \[
\begin{bmatrix}
a & b \\
0 & d
\end{bmatrix}
\] is a subspace of \(M_2(\mathbb{R})\).

Well, there are some things we have to check. Note first that the set described above, which I will call \(S\), is just the set of all upper triangular \(2 \times 2\) matrices, and is thus a subset of \(M_2(\mathbb{R})\).

The three things we need to check are:

(a) \(0 \in S\)
(b) whenever \(A\) and \(B\) are in \(S\), then \(A + B \in S\)
(c) whenever \(A \in S\) and \(c \in \mathbb{R}\), then \(cA \in S\).

- The first property is true. The elements of \(S\) are all matrices of the form \[
\begin{bmatrix}
a & b \\
0 & d
\end{bmatrix}
\] for \(a, b, d \in \mathbb{R}\), so taking \(a = b = d = 0\), we see that the \(0 \in S\) (here we are using that \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\] is the zero vector of \(M_2(\mathbb{R})\)). We could also say simply that the zero matrix in \(M_n(\mathbb{R})\) is upper triangular.

- The second property also holds. Suppose \[
\begin{bmatrix}
a_1 & b_1 \\
0 & d_1
\end{bmatrix}
\] and \[
\begin{bmatrix}
a_2 & b_2 \\
0 & d_2
\end{bmatrix}
\] are two element of \(S\). Then \[
\begin{bmatrix}
a_1 & b_1 \\
0 & d_1
\end{bmatrix} + \begin{bmatrix}
a_2 & b_2 \\
0 & d_2
\end{bmatrix} = \begin{bmatrix}
a_1 + a_2 & b_1 + b_2 \\
0 & d_1 + d_2
\end{bmatrix} \in S,
\] because it takes the right form. That is, the sum of two upper triangular matrices is upper triangular.

- Finally property three holds, because \(cA = \begin{bmatrix}
ca_1 & cb_1 \\
0 & cd_1
\end{bmatrix} \in S\) for all \(A \in S\) and \(c \in \mathbb{R}\). That is, the product of an upper triangular matrix and a scalar is still an upper triangular matrix.

Because each of the three properties above held, we can conclude that \(S\) is a subspace of \(M_2(\mathbb{R})\).

Do one of the two problems on this page. Indicate clearly which one you want me to grade.

7a. (14 pts) Let \(V\) and \(W\) be vectors spaces, \(T : V \rightarrow W\) be an onto linear transformation, and suppose that \(\{v_1, \ldots, v_k\}\) spans \(V\). Show that \(\{T(v_1), \ldots, T(v_k)\}\) spans \(W\).

Suppose that \(w \in W\). Then it is enough to show that there are \(c_1, \ldots, c_n \in \mathbb{R}\) such that \(c_1T(v_1) + \cdots + c_nT(v_n) = w\). Because \(T\) is onto, there is a \(v \in V\) such that \(T(v) = w\). We know that there exists \(c_1, \ldots, c_n \in \mathbb{R}\) such that \(c_1v_1 + \cdots + c_nv_n = v\) because \(\{v_1, \ldots, v_k\}\) spans \(V\). Acting by \(T\) on both sides of this equation yields \(w = T(v) = T(c_1v_1 + \cdots + c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n)\) because \(T\) is a linear transformation. This is exactly what we needed to show, so we are finished.

7b. (14 pts) Use the coordinate mapping to determine if \(B = \{1 - t + t^2, 4t - 4t^2, 2 + 3t^2\}\) is a basis of \(\mathbb{P}_2\).
It is enough to show that $\mathcal{B}$ is linearly independent because we know that $\dim(\mathbb{P}_2) = 3$ (we have a theorem which states that if a subset of $V$ is linearly independent and has the same size as $\dim(V)$, the it is a basis). We know from class that $\mathcal{B}$ is linearly independent if and only if $[\mathcal{B}]_\mathcal{C}$ is a basis of $\mathbb{R}^3$ where $\mathcal{C} = \{1, t, t^2\}$ is the standard basis of $\mathbb{P}_2$ (this holds because the coordinate mapping is an isomorphism). Now $[\mathcal{B}]_\mathcal{C} = \{[1 - t + t^2]_\mathcal{C}, [4t - 4t^2]_\mathcal{C}, [2 + 3t^2]_\mathcal{C}\} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 4 & 0 \\ 1 & -4 & 3 \end{bmatrix}$. We can see that this is a basis of $\mathbb{R}^3$ by forming the matrix $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 4 & 0 \\ 1 & -4 & 3 \end{bmatrix}$ and checking (via row reducing, of course) to see that it has rank 3. If so, the IMT is enough to show us that the columns are indeed linearly independent. The row reduced echelon form of $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 4 & 0 \\ 1 & -4 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which does have rank 3, so we are finished. That is, because taking coordinates is an isomorphism, $\mathcal{B}$ is a basis of $\mathbb{P}_2$. 