We label the rows $R_1$ and $R_2$.

$$\begin{bmatrix} 4 & 12 & 8 \\ 2 & 10 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 10 & 0 \end{bmatrix} R_1 \rightarrow R_1/4$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & -4 \end{bmatrix} R_2 \rightarrow R_2 - 2R_1$$

This is echelon form.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} R_2 \rightarrow R_2/4$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \end{bmatrix} R_2 \rightarrow R_1 - 3R_2$$

This is the reduced echelon form of our matrix. It corresponds to the linear system of equations

$$\begin{align*}
x_1 &= 5 \\
x_2 &= -1
\end{align*}$$

which has solution \(\{5, -1\}\).

$$\begin{bmatrix} 2 & 8 & 2 & 6 & 0 \\ 6 & 2 & 10 & 8 & 10 \\ 6 & 4 & 2 & 12 & 18 \\ 2 & 10 & 20 & 14 & 16 \end{bmatrix}$$

We label the rows, $R_1$, $R_2$, $R_3$, and $R_4$.

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 0 \\ 6 & 2 & 10 & 8 & 10 \\ 6 & 4 & 2 & 12 & 18 \\ 2 & 10 & 20 & 14 & 16 \end{bmatrix} R_1 \rightarrow R_1/2$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 0 \\ 0 & -22 & 4 & -4 & 10 \\ 6 & 4 & 2 & 12 & 18 \\ 2 & 10 & 20 & 14 & 16 \end{bmatrix} R_2 \rightarrow R_2 - 6R_1$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 0 \\ 0 & -22 & 4 & -4 & 10 \\ 0 & -20 & -4 & 0 & 18 \\ 2 & 10 & 20 & 14 & 16 \end{bmatrix} R_3 \rightarrow R_3 - 6R_1$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 0 \\ 0 & -22 & 4 & -4 & 10 \\ 0 & -20 & -4 & 0 & 18 \\ 0 & 2 & 18 & 8 & 16 \end{bmatrix} R_4 \rightarrow R_4 - 2R_1$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 0 \\ 0 & 2 & 18 & 8 & 16 \\ 0 & -20 & -4 & 0 & 18 \\ 0 & -22 & 4 & -4 & 10 \end{bmatrix}$$

Exchange $R_2$ and $R_4$.
The matrix we have at this point is in row echelon form.

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 13 \\
0 & 1 & 9 & 4 & 8 \\
0 & 0 & 1 & 0 & 18 \\
0 & 0 & 0 & 1 & 2 \\
\end{bmatrix}
R_3 \rightarrow R_3 - 80/176R_4
\]

\[
\begin{bmatrix}
1 & 4 & 1 & 3 & 0 \\
0 & 1 & 9 & 4 & 8 \\
0 & 0 & 1 & 0 & 18 \\
0 & 0 & 0 & 1 & 2 \\
\end{bmatrix}
R_2 \rightarrow R_2 - 4R_4
\]

\[
\begin{bmatrix}
1 & 4 & 1 & 3 & 0 \\
0 & 1 & 9 & 0 & 0 \\
0 & 0 & 1 & 0 & 18 \\
0 & 0 & 0 & 1 & 2 \\
\end{bmatrix}
R_2 \rightarrow R_2 - 9R_3
\]

\[
\begin{bmatrix}
1 & 4 & 1 & 0 & -213/88 \\
0 & 1 & 0 & 0 & 18/176 \\
0 & 0 & 1 & 0 & 18/176 \\
0 & 0 & 0 & 1 & 2 \\
\end{bmatrix}
R_1 \rightarrow R_1 - 4R_2
\]
This matrix is in reduced row echelon form. It corresponds to the linear system

\[
\begin{align*}
x_1 &= \frac{213}{88} \\
x_2 &= -\frac{162}{176} \\
x_3 &= \frac{18}{176} \\
x_4 &= 2
\end{align*}
\]

So the solution set to this linear system of equations is \{\frac{213}{88}, -\frac{162}{176}, \frac{18}{176}, 2\}.
Definition of an algorithm: A sequence of step by step instructions for performing some task.

Begin the algorithm with $i = 1$.

1. Locate the leftmost pivot column which has a nonzero entry in some row $R_t$, for $i \leq t \leq n$ (call this column $C$).

2. If necessary, exchange row $R_i$ with $R_t$.

3. Call $E_i$ the entry in column $C$ and row $R_i$. Replace $R_i$ with $R_i/E_i$.

4. Call $E_t$ the entry in column $C$ and row $R_t$. Then for each $t > i$, replace $R_t$ with $R_t - E_tR_i$.

5. If $i \neq n - 1$, repeat steps 1-4, beginning the algorithm with $i + 1$.

Just a fancy way of saying:

1. Move the row which has the leftmost nonzero entry to the top.

2. Divide by that non-zero entry to get its leading entry to be a 1.

3. Use that leading 1 to get rid of all the leading entries in the rows below.

4. Start over, doing steps 1-3 on the submatrix you get ignoring the first row.

Algorithm for turning a row echelon matrix into a reduced row echelon matrix.

1. Locate the rightmost pivot column which consists of more than just one non-zero entry or whose pivot is not one. Call this column $C$.

2. Locate the row such that the leading entry in this row is in column $C$. If this is row $R_j$, we call the leading entry $E_j$. Replace $R_j$ with $R_j/E_j$.

3. Call $E_i$ the entry in column $C$ and row $R_i$ for $i < j$. Then for each $i < j$, replace $R_i$ with $R_i - E_iR_j$.

4. If $C$ is not the leftmost non-zero column, repeat steps 1-4.

This is just a fancy way of saying

1. Find the rightmost pivot column which consists of more than just a leading 1.

2. If the entry in that column isn’t 1, divide so that it is.

3. Use that entry to make all the entries above it in the matrix zero.

4. Keep doing this until you are in reduced row echelon form.

Theorem: Each matrix is row equivalent to one and only one reduced echelon matrix. That means that the reduced row echelon form of a matrix is unique.

A given matrix is row equivalent to many row echelon matrices, however.

$$
\begin{bmatrix}
1 & 1 & 2 & 1 \\
1 & 2 & 1 & 3 \\
0 & 1 & -1 & 3
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 1 & 2 & 1 \\
0 & 1 & -1 & 2 \\
0 & 1 & -1 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 2 & 1 \\
0 & 1 & -1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

So ... that means $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$? This linear system is obviously inconsistent.

$$
\begin{bmatrix}
1 & 1 & 2 & 1 \\
1 & 2 & 1 & 3 \\
0 & 1 & -1 & 2
\end{bmatrix}
$$
\[
\begin{bmatrix}
1 & 1 & 2 & 1 \\
0 & 1 & -1 & 2 \\
0 & 1 & -1 & 2 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 2 & 1 \\
0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 1 & 2 & 1 \\
0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 3 & -1 \\
0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

So it corresponds to the linear system
\[
\begin{align*}
x_1 + 3x_3 &= -1 \\
x_2 - x_3 &= 2.
\end{align*}
\]

We can see that there are infinitely many solutions.

Anything that looks like \( \{s_1, s_2, s_3\} \) will work as long as \( s_1 = -1 - 3s_3 \) and \( s_2 = 2 + s_3 \).

We call \( x_3 \) a free variable in this circumstance (the other variables are called basic variables).

We may describe the solution as
\[
\begin{align*}
x_1 &= -1 - 3x_3 \\
x_2 &= 2 + x_3 \\
x_3 &= \text{free}
\end{align*}
\]

Another form would be to say that the solution set is \( \{-1 - 3x_3, 2 + x_3, x_3\} \).

We call the type of solution given above a general solution, because it gives an explicit description of all possible solutions.

Note that an augmented matrix has free variables if and only if the coefficient matrix has more columns than pivot columns. Let’s call any column which isn’t a pivot column a free column.

Fact: If there are no free columns, then the matrix (in reduced row echelon form) looks like
\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & b_1 \\
0 & 1 & \cdots & 0 & b_2 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & b_n \\
\end{bmatrix}
\]

With such a matrix, we know that there will be exactly one solution.

Fact: If a matrix is consistent, and has a free column, then there are infinitely many solutions (free columns give free variables, and free variables give infinitely many solutions).

Fact: An augmented matrix is consistent if and only if the rightmost column is not a pivot column (that is, there is not a row of the form \([0, \cdots, 0, b]\) for \( b \neq 0 \)). If such a row exists, then it implies that
\[
0 \cdot x_1 + \cdots + 0 \cdot x_n = b,
\]
which obviously can’t happen. If no such row exists then each basic variable can be described as a constant plus the sum of free variables.

These three facts imply the statement made yesterday that there are only 3 possible outcomes for a system of linear equations: no solutions, exactly one solution, or infinitely many solutions.