How much price changes as interest rates change

- Initial + shifted rate curve
- Computing prices is straightforward
- Defining changes in interest rates

Interest Rate Factor

- An interest rate factor is a random variable that impacts interest rates in some way.
- One factor driving all interest rates
- Two or more factors.
- Factor itself is an interest rate.
- Factor is not interest rate.
DV01

- \( P(y) \): price-rate function of a fixed-income security, where \( y \) is an interest rate factor.
- For example: the term structure of interest rates is flat at 2.77% and the rates move up and down in parallel.

In this environment

- As of May 28, 2010
- US Treasury note futures: TYU0.
- Call option on TYU0: strike at 120, maturity of Aug. 27, 2010
- This option gives its owner the right to purchase TYU0 at the strike price 120.

Price-rate function of TYU0
DV01: dollar value of an '01 (i.e., .01%)

- The change in the value (price) of a fixed income security for a one-basis point decline in rates.

\[
DV01 \equiv - \frac{\Delta P}{10,000 \times \Delta y}
\]

- \((\Delta P)/(\Delta y)\) ==> Slope of the line ==> Use points relatively close ==> In the limit, the slope of the line tangent to the price-rate curve at the desired rate level ==> derivative

\[
DV01 \equiv - \frac{1}{10,000} \frac{dP(y)}{dy}
\]
This chapter's definition: very general.

<table>
<thead>
<tr>
<th>7-Year Par Rate</th>
<th>TYU0</th>
<th>DY01</th>
<th>TYUC 120</th>
<th>DY01</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.72%</td>
<td>120.0780</td>
<td>1.9194</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.77%</td>
<td>119.7061</td>
<td>.07442</td>
<td>1.7383</td>
<td>.03505</td>
</tr>
<tr>
<td>2.82%</td>
<td>119.3338</td>
<td>1.5689</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- DV01 at 2.77%

\[
\frac{\Delta P}{10,000 \times \Delta y} = \frac{119.3338 - 120.0780}{10,000 \times (2.82\% - 2.72\%)} = .07442
\]

- This chapter's definition: very general.
A hedging example, part I: Hedging a call option

- If a market maker sells $100 million face value of the call option and rates are at 2.77%, how might the market maker hedge interest rate exposure by trading in the underlying TYU0?
- Buy or sell the TYU0?

Duration

- Duration measures the percentage change in the value of a security for a unit change in rates (10,000 basis points).

\[ D = -\frac{1}{P} \frac{\Delta P}{\Delta y} \]
\[ D = \frac{1}{\frac{dP}{dy}} \]
\[ D = -\frac{(119.3338 - 120.0780)/119.7061}{2.82\% - 2.72\%} = 6.217 \]
\[ \frac{\Delta P}{P} = -D\Delta y \]
\[ \Delta P = (-6.217 \times 0.0001) \times 119.7061 = -0.07442 \]

**TABLE 4.2**  Selected Model Prices and Durations for TYU0 and TYU0C 120 as of May 28, 2010

<table>
<thead>
<tr>
<th>7-Year Par Rate</th>
<th>TYU0</th>
<th>TYU0C</th>
<th>Duration</th>
<th>120</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.72%</td>
<td>120.0780</td>
<td>1.9194</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.77%</td>
<td>119.7061</td>
<td>6.217</td>
<td>1.7383</td>
<td>201.6</td>
<td></td>
</tr>
<tr>
<td>2.82%</td>
<td>119.3338</td>
<td>1.5689</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Duration Terminology**

- Effective duration:
  duration computed for any assumed change in the term structure of interest rates.
- Macaulay duration or modified duration:
  interest rate sensitivity with respect to a change in yield-to-maturity.
Convexity

- Convexity: how interest rate sensitivity changes with rates

\[ C = \frac{1}{P} \frac{d^2P}{dy^2} \]
\[
\frac{127.172545 - 127.552549}{1.77\% - 1.72\%} = -760.008
\]
\[
\frac{\Delta^2 P}{\Delta y^2} = \frac{-757.956 + 760.008}{1.795\% - 1.745\%} = 4104
\]
\[
C = \frac{1}{P} \frac{\Delta^2 P}{\Delta y^2} = \frac{4104}{127.172545} = 32.3
\]

**Measuring the price sensitivity of portfolios**

\[P = \sum P_i\]
\[dP = \sum dP_i\]
\[\frac{1}{10,000} \frac{dP}{dy} = \sum \frac{1}{10,000} \frac{dP_i}{dy}\]
\[DV'01 = \sum DV'01_i\]

\[-\frac{1}{P} \frac{dP}{dy} = \sum -\frac{1}{P} \frac{dP_i}{dy}\]
\[-\frac{1}{P} \frac{dP}{dy} = \sum -\frac{P_i}{P} \frac{1}{P} \frac{dP_i}{dy}\]
\[D = \sum \frac{P_i}{P} D_i\]
\[C = \sum \frac{P_i}{P} C_i\]
A hedging example: the negative convexity of callable bonds

- A callable bond is a bond that the issuer may repurchase or call at some fixed set of prices on some fixed set of dates
- Price = (price of bond) - (value of embedded option)

**TABLE 5.1** Selected Option Prices, Underlying Bond Prices, and DV01s at Various Rate Levels

<table>
<thead>
<tr>
<th>Rate Level</th>
<th>Option Price</th>
<th>Option DV01</th>
<th>Bond Price</th>
<th>Bond DV01</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.99%</td>
<td>8.2148</td>
<td>108.2615</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.00%</td>
<td>8.1506</td>
<td>0.0641</td>
<td>108.1757</td>
<td>0.0657</td>
</tr>
<tr>
<td>4.01%</td>
<td>8.0866</td>
<td></td>
<td>108.0901</td>
<td></td>
</tr>
<tr>
<td>4.99%</td>
<td>3.0871</td>
<td></td>
<td>100.0780</td>
<td></td>
</tr>
<tr>
<td>5.00%</td>
<td>3.0501</td>
<td>0.0399</td>
<td>100.0000</td>
<td>0.0779</td>
</tr>
<tr>
<td>5.01%</td>
<td>3.0134</td>
<td></td>
<td>99.9221</td>
<td></td>
</tr>
<tr>
<td>5.99%</td>
<td>0.7993</td>
<td></td>
<td>92.9322</td>
<td></td>
</tr>
<tr>
<td>6.00%</td>
<td>0.6979</td>
<td>0.0124</td>
<td>92.5613</td>
<td>0.0709</td>
</tr>
<tr>
<td>6.01%</td>
<td>0.6796</td>
<td></td>
<td>92.4003</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 5.2** Selected Option Prices, Underlying Bond Prices, and Durations at Various Rate Levels

<table>
<thead>
<tr>
<th>Rate Level</th>
<th>Option Price</th>
<th>Option Duration</th>
<th>Bond Price</th>
<th>Bond Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.99%</td>
<td>8.2148</td>
<td>108.2615</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.00%</td>
<td>8.1506</td>
<td>78.60</td>
<td>108.1757</td>
<td>7.92</td>
</tr>
<tr>
<td>4.01%</td>
<td>8.0866</td>
<td></td>
<td>108.0901</td>
<td></td>
</tr>
<tr>
<td>4.99%</td>
<td>3.0871</td>
<td></td>
<td>100.0780</td>
<td></td>
</tr>
<tr>
<td>5.00%</td>
<td>3.0501</td>
<td>120.82</td>
<td>100.0000</td>
<td>7.79</td>
</tr>
<tr>
<td>5.01%</td>
<td>3.0134</td>
<td></td>
<td>99.9221</td>
<td></td>
</tr>
<tr>
<td>5.99%</td>
<td>0.7993</td>
<td></td>
<td>92.9322</td>
<td></td>
</tr>
<tr>
<td>6.00%</td>
<td>0.6979</td>
<td>179.70</td>
<td>92.5613</td>
<td>7.67</td>
</tr>
<tr>
<td>6.01%</td>
<td>0.6796</td>
<td></td>
<td>92.4003</td>
<td></td>
</tr>
</tbody>
</table>
At 5%, callable bond price = 100 - 3.0501 = 96.9499
DV01 = 0.0779 - 0.0369 = 0.0410
Convexity: -223 = 103.15% x 73.63 - 3.15% x 9503.33

First-order approximation

\[ \frac{\Delta P}{P} \approx -D \Delta y \]
Estimating price changes and returns with DV01, duration, and convexity

\[ P(y + \Delta y) \approx P(y) + \frac{dP}{dy} \Delta y + \frac{1}{2} \frac{d^2P}{dy^2} \Delta y^2 \]

\[ \frac{\Delta P}{P} \approx \frac{1}{2} \frac{d^2P}{dy^2} \Delta y^2 + \frac{1}{2} \frac{d^2P}{dy^2} \Delta y^2 \]

\[ \frac{\Delta P}{P} \approx -D\Delta y + \frac{1}{2} C\Delta y^2 \]

\[ P(2.50\%) = P(2.77\%) - PD\Delta y + (1/2)PC\Delta y^2 \]

\[ = 1.738 - 350.4\% \times (-.27\%) + \frac{1}{2} \times 46682.7\% \times (-.27\%)^2 \]

\[ = 2.854 \]

TYUOC 120 at 2.77%
Price = 1.738
C = 26860
D = 201.6
PC = 46682.7
PC = 350.4
### Measures of price sensitivity based on parallel yield shifts

- General, one-factor framework.
- Measures of price sensitivity in a more restricted setting, namely, that of parallel shifts in yield.

### Yield-based DV01

- Assumes that the yield of a security is the interest rate factor

\[
P(y) = \sum_{t=1}^{T} \frac{100c/2}{(1+y/2)^t} + \frac{100}{(1+y/2)^{2T}}
\]

\[
DV01 = \frac{1}{10,000} \frac{dP}{dy}
\]

\[
dP \frac{dy}{dy} = \frac{1}{1+y/2} \left[ \sum_{t=1}^{T} t \frac{100c/2}{(1+y/2)^t} + T \frac{100}{(1+y/2)^{2T}} \right]
\]

\[
DV01 = \frac{1}{10,000} \times \frac{1}{1+y/2} \left[ \sum_{t=1}^{T} t \frac{100c/2}{(1+y/2)^t} + T \frac{100}{(1+y/2)^{2T}} \right]
\]
Modified duration: when a bond is priced using its yield, i.e.

\[ P(y) = \sum_{t=1}^{2T} \frac{100c}{2} \left( 1 + \frac{y}{2} \right)^{-t} + \frac{100}{(1 + \frac{y}{2})^{2T}} \]

\[ D_{\text{mod}} = \frac{1}{P} \times \left( \frac{1}{1 + \frac{y}{2}} \sum_{t=1}^{2T} \frac{t}{2} \frac{100c}{2} \right) \left( 1 + \frac{y}{2} \right)^{-t} + T \frac{100}{(1 + \frac{y}{2})^{2T}} \]

Macaulay duration

\[ D_{\text{Mac}} = \left( 1 + \frac{y}{2} \right) D_{\text{Mod}} \]

\[ D_{\text{Mac}} = \frac{1}{P} \left[ \sum_{t=1}^{2T} \frac{t}{2} \frac{100c}{2} \left( 1 + \frac{y}{2} \right)^{-t} + T \frac{100}{(1 + \frac{y}{2})^{2T}} \right] \]

<table>
<thead>
<tr>
<th>Date</th>
<th>Term</th>
<th>Cash Flow</th>
<th>Present Value</th>
<th>Time-Weighted PV</th>
<th>% of Wtd Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>11/30/10</td>
<td>0.5</td>
<td>1.0625</td>
<td>1.0515</td>
<td>.5258</td>
<td>.1%</td>
</tr>
<tr>
<td>5/31/11</td>
<td>1.0</td>
<td>1.0625</td>
<td>1.0406</td>
<td>1.0406</td>
<td>.2%</td>
</tr>
<tr>
<td>11/30/11</td>
<td>1.5</td>
<td>1.0625</td>
<td>1.0398</td>
<td>1.3448</td>
<td>.3%</td>
</tr>
<tr>
<td>5/31/12</td>
<td>2.0</td>
<td>1.0625</td>
<td>1.0392</td>
<td>2.0384</td>
<td>.4%</td>
</tr>
<tr>
<td>11/30/12</td>
<td>2.5</td>
<td>1.0625</td>
<td>1.0386</td>
<td>2.5236</td>
<td>.5%</td>
</tr>
<tr>
<td>5/31/13</td>
<td>3.0</td>
<td>1.0625</td>
<td>.9982</td>
<td>2.9946</td>
<td>.6%</td>
</tr>
<tr>
<td>11/30/13</td>
<td>3.5</td>
<td>1.0625</td>
<td>.9879</td>
<td>3.4375</td>
<td>.7%</td>
</tr>
<tr>
<td>5/31/14</td>
<td>4.0</td>
<td>1.0625</td>
<td>.9776</td>
<td>3.9105</td>
<td>.8%</td>
</tr>
<tr>
<td>11/30/14</td>
<td>4.5</td>
<td>1.0625</td>
<td>.9675</td>
<td>4.3338</td>
<td>.9%</td>
</tr>
<tr>
<td>5/31/15</td>
<td>5.0</td>
<td>101.0625</td>
<td>91.0749</td>
<td>455.3746</td>
<td>95.3%</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>100.1559</td>
<td>477.7621</td>
<td></td>
</tr>
</tbody>
</table>

DV01  : .04738
Duration   : 4.7208
Zero coupon bonds and duration

- Macaulay duration of a T-year zero coupon bond equals T.

\[ D_{Mac} \bigg|_{c=0} = T \]

Yield-based convexity of zero-coupon bonds

\[ C \bigg|_{y=0} = \frac{T(T + .5)}{(1 + y/2)^2} \]

- Longer-maturity zeros have greater convexity