Effective Annual Rates and Compounding

- Quoted as “5% per annum compounded semiannually”, or “5% compounded semiannually”.
- Means paying 2.5% every six months.

Semiannual Compounding

- Investing $100 at an annual rate of 5% compounded semiannually for six months generates:
  \[ 100 \times (1 + 0.05/2) = 102.50 \]
- Investing $100 at the same rate for one year instead generates:
  \[ 100 \times (1 + 0.05/2) \times (1 + 0.05/2) = 105.0625 \]
In general

- Investing $x$ at an annual rate of $r$ compounded semiannually for $T$ years generates
  
  $$x\left(1+\frac{r}{2}\right)^{2T}$$

Semiannually compounded holding period return

- What is the semiannually compounded return from investing $x$ for $T$ years and having $W$ at the end?
  
  $$r = 2 \left[ \frac{W}{x} \left( \frac{1}{2} \right)^{2T} - 1 \right]$$

Spot Rates

- The spot rate is the rate on a spot loan.
- It can vary across different maturities.
- The $t$-year spot rate is denoted $r(t)$.
- Unless otherwise specified, assume semiannual compounding frequency.
\[ r = 2 \left[ \left( \frac{1}{d(t)} \right)^{\frac{1}{2r}} - 1 \right] \]

\[ d(t) = \frac{1}{(1 + \hat{r}(t))^{2r}} \]
Table 2.1  Discount Factors, Spot Rates, and Forward Rates
Implied by Par USD Swap Rates as of May 28, 2010

<table>
<thead>
<tr>
<th>Term in Years</th>
<th>Swap Rate</th>
<th>Discount Factor</th>
<th>Spot Rate</th>
<th>Forward Rate</th>
</tr>
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<tbody>
<tr>
<td>0.5</td>
<td>.705%</td>
<td>.996489</td>
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</tr>
<tr>
<td>1.0</td>
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<td>.991306</td>
<td>.873%</td>
<td>1.046%</td>
</tr>
<tr>
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<td>1.384%</td>
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<td>.964519</td>
<td>1.450%</td>
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</tr>
</tbody>
</table>

Discount Factors

\[(100+0.705/2)*d(0.5)=100\]

\[0.875/2*d(0.5)+(100+0.875/2)*d(1)=100\]

- Term Structure, downward-sloping or inverted, upward-sloping
Term structure of spot rates

- A five-year zero-coupon bond and a ten-year zero-coupon bond are discounted using different rates.
- Coupon bond: each payment must be discounted at a different rate, according to the term of the payment.

14.25s of 1 Year Maturity

- 108 + 31.5/32 = 7.125 \cdot d(0.5) + 107.125 \cdot d(1)
- Using \[ d(t) = \frac{1}{(1 + \hat{r}(t))^t} \]
- \[ 108 + 31.5/32 = 7.125 \cdot \left(1 + \frac{\hat{r}(0.5)}{2}\right)^{0.5} + 107.125 \cdot \left(1 + \frac{\hat{r}(1)}{2}\right)^{1} \]

Forward Rates

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Forward loan and forward rates

- A forward loan is an agreement made to lend money at some future date.
- The rate of interest on a forward loan, specified at the time of the agreement as opposed to the time of the loan, is called a forward rate.
- Example: a loan two years forward, with a term of six months (2 - 2.5 years).

Forward rates

- Define $f(t)$ to be the semiannually compounded rate earned on a six month loan $(t - 0.5)$ years forward.
- Example: $f(4.5)$ is the semiannually compounded rate on a six-month loan, four years forward (i.e., the rate is agreed upon today, the loan is made in four years, and the loan is repaid in four years and six months.)

\[ f(.5) = \tilde{r}(.5) = 5.008\% \]
\[
\left(1 + \frac{f(.5)}{2}\right) \times \left(1 + \frac{f(l)}{2}\right) = \left(1 + \frac{\tilde{r}(l)}{2}\right)^2
\]
\[
\left(1 + \frac{f(.5)}{2}\right) \times \left(1 + \frac{f(l)}{2}\right) \times \left(1 + \frac{f(1.5)}{2}\right) = \left(1 + \frac{\tilde{r}(1.5)}{2}\right)^3
\]
\[
\left(1 + \frac{f(.5)}{2}\right) \times \cdots \times \left(1 + \frac{f(t)}{2}\right) = \left(1 + \frac{\tilde{r}(t)}{2}\right)^{2t}\]
TABLE 2.1  Discount Factors, Spot Rates, and Forward Rates
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</table>

- Spot sloping upward, forward above spot
- Spot sloping downward, forward below spot

If $f(2.5)$ is above $r(2)$, then $r(2.5) > r(2)$.

\[
\left(1 + \frac{\hat{r}(t - .5)}{2}\right)^{2^{t-1}} \times \left(1 + \frac{f(t)}{2}\right) = \left(1 + \frac{\hat{r}(t)}{2}\right)^{2^{t}}
\]

\[
\left(1 + \frac{\hat{r}(2)}{2}\right)^{4} \times \left(1 + \frac{f(2.5)}{2}\right) = \left(1 + \frac{\hat{r}(2.5)}{2}\right)^{5}
\]
Bond prices can be expressed in terms of forward rates

- \(108 + \frac{31.5}{32} = 7.125 \frac{d(0.5)}{2} + 107.125 \frac{d(1)}{2}\)

\[
108 + \frac{31.5}{32} = \frac{7.125}{1 + \frac{f(0.5)}{2}} + \frac{107.125}{1 + \frac{f(1)}{2}}
\]

Maturity and Bond Price

- When are bonds of longer maturity worth more than bonds of shorter maturity, and when is the reverse true?
- Consider five imaginary 4.875% coupon bonds with terms from six months to 2.5 years. Which bond would have the greatest price?

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Price</th>
<th>Forward</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 years</td>
<td>99.935</td>
<td>5.008%</td>
</tr>
<tr>
<td>1 year</td>
<td>99.947</td>
<td>4.851%</td>
</tr>
<tr>
<td>1.5 years</td>
<td>100.012</td>
<td>4.734%</td>
</tr>
<tr>
<td>2 years</td>
<td>99.977</td>
<td>4.953%</td>
</tr>
<tr>
<td>2.5 years</td>
<td>99.971</td>
<td>4.888%</td>
</tr>
</tbody>
</table>
Linear yield interpolation

- A common but unsatisfactory technique.
- A list of bond spanning the maturity range of interest.
- Bonds best suited for this purpose are those that sell near par and those liquid enough to generate accurate price quotations.
- Construct a par yield curve (yields on par bonds) by connecting the yield of these bonds with straight line.

<table>
<thead>
<tr>
<th>Coupon</th>
<th>Maturity</th>
<th>Yield</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.500%</td>
<td>07/31/01</td>
<td>5.001%</td>
<td>100.219</td>
</tr>
<tr>
<td>5.875%</td>
<td>11/30/01</td>
<td>4.964%</td>
<td>100.688</td>
</tr>
<tr>
<td>5.625%</td>
<td>11/30/02</td>
<td>4.885%</td>
<td>101.244</td>
</tr>
<tr>
<td>5.250%</td>
<td>08/15/03</td>
<td>4.887%</td>
<td>100.844</td>
</tr>
<tr>
<td>5.875%</td>
<td>11/15/04</td>
<td>4.988%</td>
<td>102.988</td>
</tr>
<tr>
<td>6.500%</td>
<td>10/15/06</td>
<td>5.107%</td>
<td>106.766</td>
</tr>
<tr>
<td>5.500%</td>
<td>02/15/08</td>
<td>5.157%</td>
<td>101.996</td>
</tr>
<tr>
<td>6.500%</td>
<td>02/15/10</td>
<td>5.251%</td>
<td>108.871</td>
</tr>
<tr>
<td>11.250%</td>
<td>02/15/15</td>
<td>5.483%</td>
<td>155.855</td>
</tr>
<tr>
<td>6.125%</td>
<td>08/15/29</td>
<td>5.592%</td>
<td>107.551</td>
</tr>
</tbody>
</table>
Kinks on the spot rate curve, especially at about 14 years.

Beyond a certain point, like five years, spot rate curves should be concave.

If a line connecting two points on a curve is below the curve, the curve is said to be concave.

If the line is above the curve, the curve is convex.
Forward curve in the figure

- Shortcomings of the forward rate curve are obvious.
- Shortcomings:
  - discount function < spot rate curve < forward rate curve.

- On Feb 15, 2011, the yields or spot rates on 9.5-year and 10-year C-STRIPS were 5.337% and 5.385%.
- The implied 6-month rate 9.5 years forward is 6.299%.
- Were the yield on the 10-year C-STRIPS to be one basis point lower, at 5.375%, a change of less than .2%, its price would rise by less than .1%.
- The forward rate would fall by 20.1 bp to 6.098%, a change of 3.2%.
Piecewise Cubics

- To build a smooth curve, assume a functional form for the discount function, for spot rates, or for forward rates.
- For example, use a cubic polynomial:

\[ d(t) = 1 + at + bt^2 + ct^3 \]

Piecewise Cubics

- A cubic polynomial for spot rate:

\[ \hat{r}(t) = \hat{r}_0 + at + bt^2 + ct^3 \]

- Even better, a cubic polynomial for each segment of the curve, then connect all the segments smoothly:

- Piecewise cubic polynomial