

# Non-Standard Asymptotic Analysis and Non-Linear Theory of Generalized Functions

Todor D. Todorov

## Abstract

The main purpose of this project is to construct a particular differential algebra  ${}^{\rho}\mathcal{E}(\Omega)$  of generalized functions which, among other things, satisfies: (a)  ${}^{\rho}\mathcal{E}(\Omega)$  is an algebra of Colombeau type, i.e.  ${}^{\rho}\mathcal{E}(\Omega)$  contains a copy of the space  $\mathcal{D}'(\Omega)$  of Schwartz distributions (Schwartz generalized functions on an open set  $\Omega \subseteq \mathbb{R}^d$ ). (b) The set of the scalars  ${}^{\rho}\mathbb{C}$  of  ${}^{\rho}\mathcal{E}(\Omega)$  ( ${}^{\rho}\mathbb{C}$  consists of the functions in  ${}^{\rho}\mathcal{E}(\mathbb{R}^d)$  with zero gradient) is a Cantor complete algebraically closed field extension of  $\mathbb{C}$ . The latter is an essential improvement of the original J.F. Colombeau theory. The applications of the algebra  ${}^{\rho}\mathcal{E}(\Omega)$  include: solving linear partial differential equations with variable coefficients or with discontinuous coefficients, solving non-linear partial differential equations originating in mathematical physics and general relativity theory with emphasis on the shock-wave solutions. The research will lead to a series of joint articles published in refereed mathematics journals and a research monograph under agreement with Kluwer Academic Publishers. The participants of this project include several mathematicians in Austria (University of Vienna and University of Innsbruck) and a Cal Poly graduate student, Guy Burger. The most essential part of the research will be done at the University of Vienna, Austria, in the period January-July, 2006, but preliminary work is going on now at Cal Poly.

**The Mathematics Subject Classification:** 46F30, 46F10, 46S20, 35D05, 35D10, 35R05, 35L67.

**Key Words:** Generalized functions, Schwartz distribution, algebras of Colombeau type, A. Robinson's non-standard analysis, non-standard functional analysis, asymptotic analysis, generalized solution of a partial differential equation, singular solution for a partial differential equation, weak solution, shock-waves.

# 1 Narrative of the Project

1° This project belongs to a relatively new branch of functional analysis known as the **non-linear theory of generalized functions**. It was founded by Jean F. Colombeau in the early 1980's (*J. F. Colombeau, New Generalized Functions and Multiplication of Distributions, North-Holland, 1984*). Colombeau constructed a commutative and associative differential algebra  $(\mathcal{G}(\Omega), +, \bullet, \partial^\alpha)$  endowed with a canonical (explicit) embedding  $\iota : \mathcal{D}'(\Omega) \rightarrow \mathcal{G}(\Omega)$  of the space of Schwartz distributions  $\mathcal{D}'(\Omega)$  (Schwartz generalized functions on an open set  $\Omega \subseteq \mathbb{R}^d$ ) into  $\mathcal{G}(\Omega)$  such that:

- (a)  $\iota(1)$  is the unit of the algebra  $\mathcal{G}(\Omega)$ .
- (b)  $\iota[\mathcal{D}'(\Omega)]$  is a differential linear subspace of  $\mathcal{G}(\Omega)$  over  $\mathbb{C}$  (a linear subspace closed under partial differentiation of any order).
- (c)  $\iota$  preserves the usual partial differentiation  $\mathcal{D}'(\Omega)$  in the sense that  $\partial^\alpha \circ \iota = \iota \circ \partial^\alpha$  on  $\mathcal{D}'(\Omega)$ .
- (d) The product  $\bullet$ , if restricted to  $\iota[C^\infty(\Omega)]$ , coincides with the usual (point-wise) product in  $C^\infty(\Omega)$  in the sense that for every  $f, g \in C^\infty(\Omega)$  we have  $\iota(fg) = \iota(f) \bullet \iota(g)$ .
- (e) The embedding  $\iota$  of  $\mathcal{D}'(\Omega)$  into  $\mathcal{G}(\Omega)$  is sheaf-preserving (it preserves the restriction on an open set). **We summarize all that as**  $C^\infty(\Omega) \subset \mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega)$ .

2° Because  $\mathcal{G}(\Omega)$  contains a copy of the Schwartz distributions, all achievements of classical functional analysis to partial differential equations (PDE) and mathematical physics are incorporated in the new theory. However this chain of embeddings solves an old and seemingly unsolvable problem known as **problem of multiplication of Schwartz distributions**: the distributions can be multiplied within the algebra  $\mathcal{G}(\Omega)$ . Following Colombeau's breakthrough, numerous similar algebras of generalized functions were constructed by Colombeau himself and other authors (including the author of this proposal Todor Todorov [1], [2], [3], [5], [7], [9], [10], [12]). These algebras of Colombeau type were applied to different areas of pure and applied mathematics, mostly to linear partial differential equations with variable coefficients and non-linear partial differential equations (M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations*, Pitman Research Notes Math., 259, Longman, Harlow, 1992). We refer to this branch of analysis as the *non-linear theory of generalized functions*. We should notice that the non-linear theory of generalized functions is honored with an exclusive code in **The Mathematics Subject Classification**: 46F30 Generalized functions for nonlinear analysis (Rosinger, Colombeau, nonstandard, etc.).

3° In spite of the overall success the non-linear theory of generalized functions has a serious weakness closely related to the properties of the scalars  $\overline{\mathbb{C}}$  of the

algebra  $\mathcal{G}(\Omega)$  ( $\overline{\mathbb{C}}$  is defined as the set of all functions in  $\mathcal{G}(\mathbb{R}^d)$  with zero gradient). Many difficulties in the theory and its applications originate from the fact that  $\overline{\mathbb{C}}$  is not a field (as any scalars should be) but it is rather a ring with zero divisors larger than  $\mathbb{C}$ . This peculiarity of the scalars hampers the progress of the theory and its applications.

The **main purpose of this project** is to correct the defect of the scalars  $\overline{\mathbb{C}}$  just described thus extending the scope of the applications of the theory while preserving all of the attractive features of the original Colombeau theory. An initial success in this direction was achieved by the author of this project (jointly with M. Oberguggenberger) in the article (*M. Oberguggenberger and T.D. Todorov* [9]), where a new algebra  ${}^\rho\mathcal{E}(\Omega)$  of generalized functions called **asymptotic functions** was constructed. The algebra  ${}^\rho\mathcal{E}(\Omega)$  is similar to but different from a typical algebra  $\mathcal{G}(\Omega)$  of Colombeau type. Still  ${}^\rho\mathcal{E}(\Omega)$  has several **important advantages compared with a Colombeau type of algebra  $\mathcal{G}(\Omega)$ . Here are some:**

(a) The set of the scalars  ${}^\rho\mathbb{C}$  of the algebra  ${}^\rho\mathcal{E}(\Omega)$  ( ${}^\rho\mathbb{C}$  is defined as the functions in  ${}^\rho\mathcal{E}(\mathbb{R}^d)$  with zero gradient) constitutes an algebraically closed Cantor complete field (in contrast to its counterpart  $\overline{\mathbb{C}}$  in Colombeau theory which is a ring with zero divisors). The elements of  ${}^\rho\mathbb{C}$  are called **asymptotic numbers** for the following reason: The field  ${}^\rho\mathbb{C}$  is isomorphic to the valuation field introduced by A. Robinson in 1970. According to a more recent result (T. Todorov and R. Wolf [12]), the field  ${}^\rho\mathbb{C}$  is also isomorphic to the field of the generalized power series with coefficients in the field of the complex non-standard numbers  ${}^*\mathbb{C}$ . Since  $\mathbb{C} \subset {}^*\mathbb{C}$ , it follows that the field of asymptotic series with complex coefficients is embedded as a subfield in  ${}^\rho\mathbb{C}$ . This explains the name “asymptotic numbers” for the elements of  ${}^\rho\mathbb{C}$ . As in Colombeau theory, the asymptotic functions in  ${}^\rho\mathcal{E}(\Omega)$  can be identified as pointwise functions with values in the field of asymptotic numbers  ${}^\rho\mathbb{C}$  (Todor Todorov [10]). The latter explains the name “asymptotic functions” for the elements of  ${}^\rho\mathcal{E}(\Omega)$ .

**Remark:** The improvement of the properties of the scalars of the generalized functions (compared with Colombeau’s theory) described above is achieved at the price of involvement of the methods of A. Robinson’s **non-standard analysis**: The field of the scalars  ${}^\rho\mathbb{C}$  is defined as a particular factor ring within the field of complex non-standard numbers  ${}^*\mathbb{C}$ , where  $\rho$  is a fixed positive infinitesimal in  ${}^*\mathbb{R}$ . Similarly, the algebra of asymptotic functions  ${}^\rho\mathcal{E}(\Omega)$  is defined as a particular factor ring within the class of non-standard functions  ${}^*\mathcal{E}(\Omega)$ , where  $\mathcal{E}(\Omega) = \mathcal{C}^\infty(\Omega)$ . The construction of  ${}^\rho\mathbb{C}$  and  ${}^\rho\mathcal{E}(\mathbb{R}^d)$  is in the spirit of A. Robinson’s asymptotic analysis (A. H. Lightstone and A. Robinson, *Nonarchimedean Fields and Asymptotic Expansions*, North-Holland, Amsterdam, 1975).

(b) The construction of the algebra of asymptotic functions  ${}^\rho\mathcal{E}(\mathbb{R}^d)$  is simpler than Colombeau’s construction in the sense that the number of regularization parameters and quantifiers is reduced (say, by two).

(c) There are several general theorems in  ${}^*\mathbb{C}$  and  ${}^*\mathcal{E}(\Omega)$ , called the Saturation Principle and the Transfer Principle, borrowed from A. Robinson's non-standard analysis. These principles are without counterparts in Colombeau theory.

(d) Saturation and transfer principles in  ${}^*\mathbb{C}$ , along with the properties of the scalars  ${}^\rho\mathbb{C}$ , make the analysis on  ${}^\rho\mathbb{C}$  better connected (compared with the original Colombeau theory) with some other branches of mathematics: **A. Robinson's non-standard analysis, A. Robinson's asymptotic analysis, model theory of fields, Maslov theory and the  $p$ -adic analysis.**

**Remark:** The theory of the asymptotic functions has some common features with  **$p$ -adic analysis** due to the fact that the field  ${}^\rho\mathbb{C}$  and the field of the  $p$ -adic numbers  $\mathbb{Q}_p$  are both ultrametric spaces. We should mention however that **the theory of asymptotic functions** has an important **advantage over  $p$ -adic analysis**: the field of the real asymptotic numbers  ${}^\rho\mathbb{R}$  is a totally ordered field in contrast to the field of the  $p$ -adic numbers  $\mathbb{Q}_p$  which is non-orderable. Consequently,  ${}^\rho\mathbb{C}$  and  ${}^\rho\mathbb{R}$  are endowed with **standard part mapping** (important for the applications) which does not have a counterpart in  $\mathbb{Q}_p$ .

4° The construction of  ${}^\rho\mathcal{E}(\Omega)$  in [9] was followed by several more articles, where the asymptotic functions were used for proving the existence results for **some partial differential equations** without solutions in the classical spaces of functions and Schwartz distributions (*T.D. Todorov* [5], [7], [10]). These (and some other M. Oberguggenberger [8]) publications attracted the attention of the experts in the field and I received an offer from Kluwer Academic Publishers (a copy of the agreement enclosed) to write - jointly with Michael Oberguggenberger - a research monograph on this approach. Kluwer also requested that an essential part of this monograph be devoted to an introduction to non-standard analysis adapted to the particular needs of the theory and accessible to the researchers working in this field. The second part of the monograph will include one (or two) chapters on applications of the new theory to partial differential equations and mathematical physics.

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