Research Statement

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1. Overview

My research is in Commutative Algebra (MSC 13D40, 13P99, 14C99, 14M06, 14M12). I am most interested in the links between Commutative Algebra and Algebraic Geometry, especially with respect to reduced 0-dimensional schemes. My work currently focuses on

(1) Hilbert functions:
 (a) Investigating the Eisenbud-Green-Harris Conjecture;
 (b) Hilbert functions of fat point subschemes with support from special configurations;
(2) Determinantal schemes.

2. Hilbert Functions

2.1. Introduction and Motivation: Macaulay's Theorem.

On a first encounter, 0-dimensional schemes may seem to be devoid of substance. However, the field has a deep history and yet still manages to host many exciting open problems [28]. The importance of 0-dimensional schemes was demonstrated when Castelnuovo showed that non-trivial data about a curve can be retrieved from the finite sets of points arising as the general hyperplane sections of the curve [13]. Many algebraic ideas have been introduced in order to obtain information about points. The Hilbert function has played a central role in many problems. We begin with some notation.

Let k be an algebraically closed field of characteristic 0 and let R denote the polynomial ring $k[x_0, \ldots, x_n]$. We say that $F \in R$ is *homogeneous* if every term of F has the same degree (e.g. $x_0^2 - 10x_1x_3$ is homogeneous of degree 2). An ideal $I \subseteq R$ is *homogeneous* if it can be generated by a set of homogeneous polynomials. We group the elements of I by degree which results in a collection of finite dimensional vector spaces. For example, let $I = (F, G)$ where $F = x_0 - x_2$ and $G = x_1^2 - x_1x_2$. Any homogeneous polynomial of degree 2 in I has the form $c_1x_0F + c_2x_1F + c_3x_2F + c_4G$, where c_1, c_2, c_3, c_4 are scalars. Thus, the dimension of the *degree two piece* I_2 of I is 4, denoted $\dim_k I_2 = 4$.

If $I \subseteq R$ is a homogeneous ideal and A is the quotient ring R/I, then we incorporate the degree-by-degree dimensions of I in a sequence called the *Hilbert function* of A, denoted $H(A)$. More precisely, $H(A)$ is the sequence $\{H(A,d)\}_{d\geq 0}$ of non-negative integers where $H(A,0) = 1$ and

$$H(A,d) = \dim_k A_d = \dim_k (R/I)_d = \dim_k R_d - \dim_k I_d = \binom{n+d}{d} - \dim_k I_d \quad \text{for } d \geq 1.$$

We define the *first difference* function as $\Delta H(A) := \{c_d\}_{d\geq 0}$, where $c_0 = 1$ and $c_d = H(A,d) - H(A,d-1)$ for $d \geq 1$. For example, if $I = (x_0 - x_2, x_1^2 - x_1x_2)$ then $H(A) = 1\ 2\ 2\ 2 \cdots$ and $\Delta H(A) = 1\ 1\ 0 \cdots$.

Hilbert functions have been extensively studied. Perhaps the most celebrated result is *Macaulay's Theorem* which implies that $H(A)$ can be described using *lex ideals*.

Definition 2.1. A *lex ideal* is a monomial ideal L which is minimally generated by $\{m_1, \ldots, m_r\}$, where, for $j = 1, \ldots, r$, all monomials of degree $\deg(m_j)$ which are larger than m_j in the degree-lexicographic ordering are contained in L.

Macaulay's Theorem [25, 32] Let $I \subseteq R$ be a homogeneous ideal. Then there exists a lex ideal L such that $H(R/I) = H(R/L)$.

To demonstrate, let $I = (x_0^2 + x_0x_1, x_0^3x_1, x_0^5 + x_1^5) \subseteq R = k[x_0, x_1]$. Then $\dim_k I_1 = 0, \dim_k I_2 = 1, \dim_k I_3 = 2, \dim_k I_4 = 4, \dim_k I_i = \dim_k R_i$ for $i \geq 5$. Let $L = L_1 + L_2 + \cdots$ be the lex ideal where L_i is generated by the largest $\dim_k I_i$ monomials in the degree-lexicographic ordering; so $L_1 = (0), L_2 = (x_0^2), L_3 = (x_0^3, x_0^2x_1), L_4 = (x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3)$, etc. Then $H(R/L) = H(R/I) = 1\ 2\ 2\ 2\ 1\ 0 \cdots$.

Much effort has gone into generalizing Macaulay's Theorem. These ideas have developed in a variety of directions. Geramita-Maroscia-Roberts [21] characterize the Hilbert functions of finite sets of distinct, reduced points in $\mathbb{P}^n$. If $\mathbb{X}$ is a set of points and $\mathbf{I}(\mathbb{X}) \subseteq R$ is the ideal consisting of all the homogeneous polynomials vanishing on $\mathbb{X}$, then the *Hilbert function of* $\mathbb{X}$ is $H(\mathbb{X}) := H(R/\mathbf{I}(\mathbb{X}))$. The Hilbert function of $\mathbb{X}$ can be exploited to obtain both algebraic data about $R/\mathbf{I}(\mathbb{X})$ and geometric information about $\mathbb{X}$.

2.2. **Ph.D. Dissertation: The Eisenbud-Green-Harris Conjecture and Points.** The Eisenbud-Green-Harris Conjecture is an attempt at generalizing Macaulay's Theorem which has gained much recent attention. It concerns ideals containing regular sequences. Technically speaking, $\{F_1, \ldots, F_n\}$ in $S := k[x_1, \ldots, x_n]$ is a *regular sequence of forms* if each F_i is a homogeneous polynomial such that F_{i+1} is not a zero-divisor on $S/(F_1, \ldots, F_i)S$ for each i. Let $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ be integers and $\mathbb{A} := \{a_1, \ldots, a_n\}$. Then $\{x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}\}$ is a regular sequence.

Definition 2.2. A *lex-plus-powers ideal with respect to* $\mathbb{A}$ is a monomial ideal L in S of the form $J + P$ where J is a lex ideal and $P = (x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})$.

Equivalently, a lex-plus-powers ideal with respect to $\mathbb{A}$ is a monomial ideal $L \subseteq S$ which is minimally generated by $\{x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}, m_1, \ldots, m_r\}$, where, for $j = 1, \ldots, r$, all monomials of degree $\deg(m_j)$ which are larger than m_j in the degree-lexicographic ordering are contained in L. For example, let $\mathbb{A} = \{2, 3, 3\}$. Then $L_1 = (x_1^2, x_2^3, x_3^3, x_1x_2^2, x_1x_2x_3)$ is a lex-plus-powers ideal with respect to $\mathbb{A}$, but $L_2 = (x_1^2, x_2^3, x_3^3, x_1x_2^2, x_1x_2x_3, x_2^2x_3)$ is not since $x_1x_3^2 >_{d-lex} x_2^2x_3$ and $x_1x_3^2$ is not in L_2.

Conjecture 2.3. [2] Let $I \subseteq S$ be a homogeneous ideal containing a regular sequence $F_1, \ldots F_n$ of forms of degrees $\deg(F_i) = a_i$. Then there exists a homogeneous ideal J containing $\{x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}\}$ such that $H(S/I) = H(S/J)$.

Clements-Lindström [3] show (in a combinatorial fashion) that for any *monomial* ideal $M \subseteq S$ containing $\{x_1^{a_1}, \ldots, x_n^{a_n}\}$ there is a lex-plus-powers ideal L with respect to $\mathbb{A}$ with $H(S/M) = H(S/L)$. Cooper-Roberts have generalized this fact.

Lemma 2.4. [4, 9] *If $I \subseteq S$ is any homogeneous ideal containing $\{x_1^{a_1}, \ldots, x_n^{a_n}\}$, then there is a lex-plus-powers ideal L with respect to $\mathbb{A}$ such that $H(S/I) = H(S/L)$.*

Thus, Conjecture 2.3 can be restated as follows.

Eisenbud-Green-Harris (EGH) Conjecture [2, 11, 12, 17] If $I \subseteq S$ is a homogeneous ideal containing a regular sequence $F_1, \ldots F_n$ of forms of degrees $\deg(F_i) = a_i$, then there is a lex-plus-powers ideal L with respect to $\mathbb{A}$ such that $H(S/I) = H(S/L)$.

If the EGH Conjecture is true, then the combinatorial work of Greene-Kleitman [23] can be used to give bounds on the growth of homogeneous ideals containing regular sequences in fixed degrees [4, 5, 9]. Using these bounds, I proved the EGH Conjecture in the following situation:

Theorem 2.5. [8] *Fix $T := k[x, y, z]$ and $T' := k[x, y]$, with $x >_{d-lex} y >_{d-lex} z$ ("$d - lex$" denotes the degree-lexicographic ordering). We also let $I := (F, G, H_1, H_2, \ldots, H_t) \subseteq T$ be a homogeneous ideal such that:*

1. *I is minimally generated by $F, G, H_1, H_2, \ldots, H_t$;*
2. $\deg(F) = \deg(G) = \deg(H_1) = \deg(H_2) = \cdots = \deg(H_t) = d$;
3. *$F, G \in T'$ is a regular sequence of forms.*

Fix $J \subseteq T$ to be the ideal generated by x^d, y^d and the t largest monomials, with respect to the degree-lexicographic ordering, in $T_d \setminus \{x^d, y^d\}$. Then $\dim_k(T_1 I_d) \geq \dim_k(T_1 J_d)$.

In addition to Theorem 2.5, and despite much effort from many commutative algebraists, the EGH Conjecture is known to be true only in some exceptional cases. The conjecture has been proven in the cases where L is an *almost complete intersection* [16], and when $n = 2$ [29]. The EGH Conjecture was originally stated in the case when each $a_i = 2$; in this case, Richert [29] has verified the conjecture for $n \leq 5$ via computer checks. Mermin-Peeva-Stillman [26] have results for ideals containing squares of the variables. Most recently, Caviglia-Maclagan [2] have proven that the EGH Conjecture is true if $a_j > \sum_{i=1}^{j-1}(a_i - 1)$ for $j = 1, \ldots, n$. On the geometric side, I [4] have proven the EGH Conjecture for Artinian reductions of ideals of reduced, distinct, finite point sets in $\mathbb{P}^2$, as well as in $\mathbb{P}^3$ under some assumptions on the degrees of the regular sequences. We now turn to this geometric setting.

Recall that a *complete intersection* is a set of points $\mathbb{Y} \subseteq \mathbb{P}^n$ such that $\mathbf{I}(\mathbb{Y})$ is generated by n homogeneous polynomials which form a regular sequence. The list $\{d_1, \ldots, d_n\}$ of the degrees of the generating homogeneous polynomials is the *type* of the complete intersection.

A finite set $\mathbb{X}$ of distinct, reduced points is obviously contained in some (in fact, many) complete intersections, if we have the freedom to pick the degrees of the defining polynomials. A main question that motivates my research is to fix these degrees and find conditions that a given Hilbert function must satisfy in order for it to be the Hilbert function of a subset of points in a complete intersection.

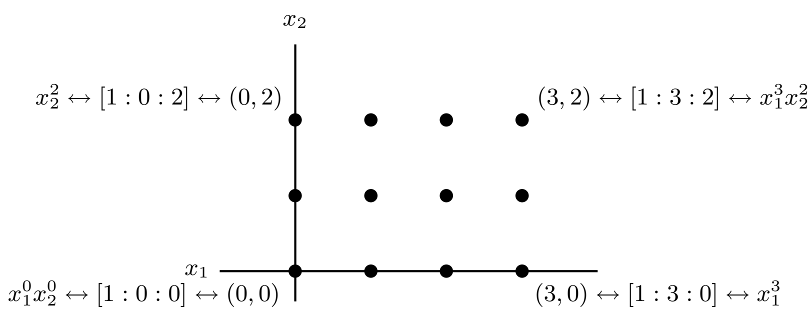
Question 2.6. Fix integers $2 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ and let $\mathcal{H}$ be the Hilbert function of some finite set of distinct points in $\mathbb{P}^n$. Do there exist finite sets of distinct, reduced points $\mathbb{X}$ and $\mathbb{Y}$ such that: (1) $\mathbb{X} \subseteq \mathbb{Y}$; (2) $H(\mathbb{X}) = \mathcal{H}$; and (3) $\mathbb{Y}$ is a complete intersection of type $\{d_1, \ldots, d_n\}$?

Question 2.6 has been answered for a family of complete intersections called *rectangular complete intersections*. If we fix integers $2 \leq d_1 \leq d_2 \leq \cdots \leq d_n$, then the set $\mathbb{Y} \subseteq \mathbb{P}^n$ of $d_1 d_2 \cdots d_n$ distinct, reduced points with integer coordinates:

$$\{[1 : a_1 : \ldots : a_n] \mid 0 \leq a_1 \leq d_n - 1, 0 \leq a_2 \leq d_{n-1} - 1, \ldots, 0 \leq a_n \leq d_1 - 1\}$$

is a *rectangular complete intersection of type* $\{d_1, \ldots, d_n\}$, and is denoted $\mathbb{Y} = \mathrm{RectCI}(d_1, \ldots, d_n)$.

Again let $R := k[x_0, x_1, \ldots, x_n]$ and $S := k[x_1, \ldots, x_n]$. Note that if $\mathbb{Y} = \mathrm{RectCI}(d_1, \ldots, d_n)$ then $A := R/(\mathbf{I}(\mathbb{Y}), x_0) = S/(x_1^{d_n}, x_2^{d_{n-1}}, \ldots, x_n^{d_1})$. So the monomials of A are in 1-1 correspondence with the points of $\mathrm{RectCI}(d_1, \ldots, d_n)$. For example, RectCI(3,4) can be visualized as the following 12 dots ($\bullet$) below with the corresponding bijections noted:



Moreover, if $\mathbb{X} \subseteq \mathbb{Y} = \mathrm{RectCI}(d_1, \ldots, d_n)$ then we have a surjection

$$R/(\mathbf{I}(\mathbb{Y}), x_0) \to R/(\mathbf{I}(\mathbb{X}), x_0),$$

where $R/(\mathbf{I}(\mathbb{X}), x_0) = S/J$ with J a monomial ideal containing $x_1^{d_n}, \ldots, x_n^{d_1}$. The above mentioned bounds of Greene-Kleitman [23] can be applied to characterize the growth of such quotients S/J. We have the immediate question:

Question 2.7. Is the characterization of Hilbert functions of subsets of non-rectangular complete intersections completely controlled by the situation prevailing in the rectangular case? That is, does $\{\mathcal{H} \mid \mathcal{H} = \Delta H(\mathbb{X}), \mathbb{X} \subseteq \mathbb{Y} \in CI(d_1, \ldots, d_n)\} = \{\mathcal{H}' \mid \mathcal{H}' = \Delta H(\mathbb{W}), \mathbb{W} \subseteq RectCI(d_1, \ldots, d_n)\}$?

In [4, 5] I show that Question 2.7 is equivalent to the EGH Conjecture restricted to this geometric setting.

Theorem 2.8. [4, 5] *Fix integers $2 \leq d_1 \leq d_2 \leq \cdots \leq d_n$. The answer to Question 2.7 is "yes" in the following cases*:

(1) $n = 2$;
(2) $n = 3$ *and either:*
   (a) $2 = d_1 \leq d_2 \leq d_3$ *or;*
   (b) $3 = d_1 \leq d_2 \leq d_3$ *or;*
   (c) $4 \leq d_1 \leq d_2 \leq d_3$ *and* $d_3 \geq d_1 + d_2 - 1$.

Interestingly enough, although independently discovered, the assumptions on the degrees of the regular sequences needed for (2c) of Theorem 2.8 are the same as that of Caviglia-Maclagan [2]. Note, however, that cases (2a) and (2b) are not completely covered by their results.

2.3. **Ongoing Research with Fat Point Subschemes.**
Many people have been interested in generalizing Macaulay's Theorem to *fat points*. It is straightforward to verify that if $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$ is a finite set of distinct, reduced points, then $\mathbf{I}(\mathbb{X})$ is the intersection of s prime ideals, namely $\mathbf{I}(\mathbb{X}) = \mathbf{I}(P_1) \cap \mathbf{I}(P_2) \cap \cdots \cap \mathbf{I}(P_s)$. A *fat point subscheme* $\mathbb{Y}$ of $\mathbb{P}^n$ is defined by a homogeneous ideal $I \subseteq R = K[x_0, \ldots, x_n]$ of the form

$$I = \mathbf{I}(P_1)^{m_1} \cap \mathbf{I}(P_2)^{m_2} \cap \cdots \cap \mathbf{I}(P_s)^{m_s},$$

where $P_1, \ldots, P_s$ (i.e. the *support* of $\mathbb{Y}$) are distinct points of $\mathbb{P}^n$ and $m_1, \ldots, m_s$ are non-negative integers. The integers $m_1, \ldots, m_s$ are called the *multiplicities* of $\mathbb{Y}$. We denote $\mathbb{X}$ as a formal sum $m_1P_1 + \cdots + m_sP_s$.

Question 2.9. Let $\mathbb{X} \subseteq RectCI(d_1, d_2)$. What are the Hilbert functions that arise from adding multiplicities to the points in $\mathbb{X}$?

A central obstacle to answering this question is that it is not known what the Hilbert functions are for fat points in general. However, if the number of points is 8 or less then we can write down all of the possible Hilbert functions for any given set of multiplicities [18, 24]. B. Harbourne (University of Nebraska - Lincoln), Z. Teitler (Texas A & M University) and myself are currently collaborating on fat points. To describe our results we introduce some notation.

If $\mathbb{Y} = \{P_1, \ldots, P_n\} \subseteq \mathbb{P}^2$ is a set of distinct, reduced points then we denote the set of fat points defined by $\cap \mathbf{I}(P_i)^a$ by $a\mathbb{Y}$. Thus $2\mathbb{Y}$ is the scheme defined by the symbolic square $\mathbf{I}(\mathbb{Y})^{(2)} = \cap \mathbf{I}(P_i)^2$. Geramita-Migliore-Sabourin [22] consider the double points $2\mathbb{Y}$ for sets $\mathbb{Y}$ in special arrangements called linear configurations.

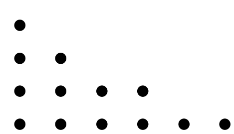
Definition 2.10. Let $0 < m_1 < \cdots < m_r$ be integers. A *linear configuration* $\mathbb{Y}$ of points in $\mathbb{P}^2$ of type $\lambda(\mathbb{Y}) = (m_1, \ldots, m_r)$ is a disjoint union $\mathbb{Y} = \mathbb{Y}_1 \cup \cdots \cup \mathbb{Y}_r \subseteq \mathbb{P}^2$ of finite subsets $\mathbb{Y}_i \subseteq \mathbb{L}_i$, where $|\mathbb{Y}_i| = m_i$ and $\mathbb{L}_1, \ldots, \mathbb{L}_r$ are distinct lines such that no point of $\mathbb{Y}$ is where any two of the lines meet.

Definition 2.11. Suppose the characteristic of the field K is zero. We define a finite set of points $\mathbb{S} \subseteq \mathbb{P}^2$ to be *standard* if each point of $\mathbb{S}$ has integer coordinates $[i : j : 1]$, such that

$$[i : j : 1] \in \mathbb{S} \implies [i' : j' : 1] \in \mathbb{S}$$

for all $[i' : j' : 1]$ with $0 \leq i' \leq i$ and $0 \leq j' \leq j$.

We see that a set of standard points are lattice points in the first quadrant, with longer rows lower, and all rows left registered. If no two rows have the same number of points, then the set of points is a *standard linear configuration.* For example, the following set of points ($\bullet$) is a standard linear configuration:



Geramita-Migliore-Sabourin [22] describe the first difference Hilbert functions of standard linear configurations. To explain this we need to describe a diagonal count operator. We do this via an example; let $v = (1, 2, 4, 6)$ (this is the vector giving the number of points on each row, starting from the top, in the above standard linear configuration) then $diag(v) = (1, 2, 3, 4, 2, 1)$ (this is the vector giving the number of points on each diagonal of slope -1, starting from the leftmost position, in the above standard linear configuration).

Suppose the characteristic of K is 0. If $\mathbb{X}$ is a finite set of distinct, reduced points in $\mathbb{P}^2$ which comprise a standard set whose row count vector from the top row to the bottom row is $m = (m_1, \ldots, m_r)$, then $\Delta H(\mathbb{X}) = diag(m)$ [22]. In fact, $\mathcal{H}$ is the first difference Hilbert function for some finite set of distinct, reduced points in $\mathbb{P}^2$ if and only if $\Delta\mathcal{H} = diag((m_1, \ldots, m_r))$ for some finite increasing positive sequence of integers m_i. Therefore, in characteristic 0, for any $\mathbb{X}'$ there is a standard linear configuration $\mathbb{X}$ such that $H(\mathbb{X}) = H(\mathbb{X}')$.

Geramita-Migliore-Sabourin [22] give a numerical criterion for when $\lambda(\mathbb{Y})$ determines $H(2\mathbb{Y})$ when $\mathbb{Y}$ is a linear configuration and the characteristic of K is 0. Cooper-Harbourne-Teitler [7] generalize this result. Our work applies in all characteristics and to arrangements more general than linear configurations. Moreover, our work applies to fat points $a\mathbb{Y}$ where $a > 2$ and even for fat points of mixed multiplicities. We use an argument similar in spirit to Bézout's Theorem. Our results concern line count configurations.

Definition 2.12. A *line count configuration* $\mathbb{Y}$ of points in $\mathbb{P}^2$ of type $\lambda(\mathbb{Y}) = (m_1, \dots, m_r)$ is a disjoint union $\mathbb{Y} = \mathbb{Y}_1 \cup \dots \cup \mathbb{Y}_r \subseteq \mathbb{P}^2$ of finite subsets $\mathbb{Y}_i \subseteq \mathbb{L}_i$, where $|\mathbb{Y}_i| = m_i$ and $\mathbb{L}_1, \dots, \mathbb{L}_r$ are distinct lines such that no point of $\mathbb{Y}$ is where any two of the lines meet.

Note that, unlike the definition of a linear configuration in [22], we do not require that the m_i be distinct or be nondecreasing in Definition 2.12. Given a line count configuration as above and a nonnegative integer vector $a = (a_1, \dots, a_r)$, we call $A = a_1\mathbb{Y}_1 + \dots + a_r\mathbb{Y}_r$ the fat point scheme associated to $\mathbb{Y}$ and a. Denote $a(A) = (a_1, \dots, a_r)$ and $\lambda(A) = (m_1, \dots, m_r)$. The goal of [7] is to give a numerical criterion for $H(A)$ to be completely determined by $a(A)$ and $\lambda(A)$. We now describe this criterion.

Given an integer vector $b = (b_1, \dots, b_l)$, we denote by $\pi(b)$ the vector whose entries are the entries of b permuted to be nondecreasing, and let $\Delta b = (b_1, b_2 - b_1, \dots, b_l - b_{l-1})$. Further, if $a = (a_1, \dots, a_r)$ and $m = (m_1, \dots, m_r)$ are positive integer vectors then we let

$$w = (m_1, 2m_1, \dots, a_1m_1, m_2, 2m_2, \dots, a_2m_2, \dots, m_r, 2m_r, \dots, a_rm_r)$$

and let

$$a \star m = \pi(w).$$

We say that $a \star m$ is *Bézout* if whenever there are two zero entries of $\Delta(a \star m)$ there is an entry between them which is strictly greater than 1. For example, if $a = (1, 3, 2)$ and $m = (2, 1, 2)$, then $w = (2, 1, 2, 3, 2, 4), a \star m = (1, 2, 2, 2, 3, 4)$ and $\Delta(a \star m) = (1, 1, 0, 0, 1, 1)$; the two consecutive zeros means that $a \star m$ is not Bézout. Note that the Property 3.2 of [22] is the same as the condition that $a \star m$ is Bézout when $a_i = 2$ for all i and $m_1 < m_2 < \dots < m_r$.

Theorem 2.13. [7] *Given positive integer vectors $a = (a_1, \dots, a_r)$ and $m = (m_1, \dots, m_r)$, let $\mathbb{Y}$ be any line count configuration with $\lambda(\mathbb{Y}) = m$ and let A be the fat point scheme associated to $\mathbb{Y}$ and a. If $a \star m$ is Bézout, then $\Delta H(A) = diag(a \star m)$.*

Theorem 2.13 raises the following question.

Question 2.14. If $a \star m$ is not Bézout, must there be line count configurations $\mathbb{Y}$ and $\mathbb{Y}'$ with $\lambda(\mathbb{Y}) = m = \lambda(\mathbb{Y}')$ and schemes A associated to $\mathbb{Y}$ and a, and A' associated to $\mathbb{Y}'$ and a, such that $H(A) \neq H(A')$?

3. Current and Future Research

I plan to continue my work with Hilbert functions. In addition, I will continue my post-doctoral research on determinantal schemes. There are five main questions motivating my current and future research programs. We briefly discuss these below.

(1) The Eisenbud-Green-Harris Conjecture was originally stated in the case when the degrees of the regular sequence elements are all 2. Theorem 2.8 considers this case in the geometric setting. I am currently collaborating with L. O'Carroll (University of Edinburgh) on this special case in the algebraic setting.

(2) Essential parts of the proofs on the EGH Conjecture from my Ph.D. dissertation [4, 8] rely on algebraic and geometric consequences for when Hilbert functions attain the largest possible growth as described by Macaulay's Theorem [1]. I am interested in generalizing this school of thought to the Hilbert functions of subsets of complete intersections and the bounds described by Greene-Kleitman [23]. In particular, many people have attempted to generalize the Hilbert functions of point sets which

have the *Cayley-Bacharach Property* (CBP); a set of s distinct points in $\mathbb{P}^n$ has the CBP if every subset of $s-1$ points has the same Hilbert function. One application in my Ph.D. dissertation [4] was to produce a family of subsets of complete intersections that are guaranteed to have the CBP. I will continue to investigate such applications.

(3) Linear configurations are special cases of more well-known k-configurations. It is known that given any Hilbert function $\mathcal{H}$ of a reduced zero-dimensional subscheme of $\mathbb{P}^n$, there is a k-configuration (which is also a reduced subscheme of $\mathbb{P}^n$) whose Hilbert function is $\mathcal{H}$ [19]. Thus it is natural to try to classify the Hilbert functions of the fat points whose support is a k-configuration. Indeed this motivates [22]. Sabourin [30, 31] generalizes k-configurations. Rather than using lines and hyperplanes to build up the reduced points, she uses complete intersections. One sees that the generalization of linear configurations to line count configurations is done in the same spirit. Indeed, we can view a line count configuration as being built up of the union of rectangular complete intersections.

Question 3.1. What are the Hilbert functions of fat point schemes whose support is the union of complete intersections?

The results of Cooper-Harbourne-Teitler [7] seem to be an initial investigation of Question 3.1. Knowing that k-configurations play such a deep role in classifying Hilbert functions of reduced points, it seems that our approach is key in characterizing the Hilbert functions of fat point schemes.

(4) One tool that I used in proving some cases of the EGH Conjecture in my Ph.D. dissertation [4] is the behavior of Hilbert functions under *liaison.*

Theorem 3.2. [2, 10, 27] *Let $J = (F_1, \dots, F_n)$ be an ideal in $S = k[x_1, \dots, x_n]$ generated by a regular sequence with $\deg(F_i) = a_i$. Let I be an ideal containing J and let $s = \sum_{i=1}^{n}(a_i - 1)$. Then*

$$H(S/J, t) = H(S/I, t) + H(S/(J : I), s - t)$$

for $0 \le t \le s$.

In the 0-dimensional setting, if $\mathbb{X}$ is a subset of a complete intersection $\mathbb{Y}$ of type $\{a_1, \dots, a_n\}$ then Theorem 3.2 gives a formula relating $\Delta H(\mathbb{X}), \Delta H(\mathbb{Y})$ and $\Delta H(\mathbb{Y} - \mathbb{X})$.

Question 3.3. Is there an analog to Theorem 3.2 for fat points?

It is likely that an answer to Question 3.3 will play an important role in studying fat points arising from rectangular complete intersections. Indeed, the original goal of the joint work Cooper-Harbourne-Teitler [7] was to make significant progress in this direction. We started by focusing on double points with support in $\mathbb{P}^2$. While doing so we consider special schemes $[2]\mathbb{X}$. If P is a point in $\mathbb{P}^2$, then we define $I([2]P)$ to be the ideal generated by the squares of the generators of $\mathbf{I}(P)$. For any finite set of points $\mathbb{X} \subseteq \mathbb{P}^2$, none of which are on the line $z = 0$, we define $[2]\mathbb{X}$ to be the scheme whose ideal is $\cap_{P \in \mathbb{X}} I([2]P)$. At first encounter the schemes $[2]\mathbb{X}$ seem a bit mysterious and unrelated to double points. However, not only are these schemes interesting in their own right but, under assumption on the arrangement of the support $\mathbb{X}$, it is easy to describe $\Delta H([2]\mathbb{X})$ and apply Theorem 3.2 to obtain a formula relating $\Delta H([2]\mathbb{X}), \Delta H([2]\mathbb{Y})$ and $\Delta H([2]\mathbb{T})$, where $\mathbb{X}$ and $\mathbb{Y}$ are disjoint subsets of $\mathbb{T}$ with $\mathbb{X} \cup \mathbb{Y} = \mathbb{T}$. As a consequence, we are able to relate $\Delta H([2]\mathbb{T})$ and the first difference Hilbert function of the double point scheme with support $\mathbb{T}$.

(5) *Postdoctoral & Additional Current Research: The Gale Transform & Determinantal Schemes*
The Gale transform takes a sufficiently nice set Γ of $r+s+2$ points in the projective space $\mathbb{P}^r$ to a set Γ' of the same number of points in $\mathbb{P}^s$. In [14, 15] Eisenbud and Popescu give a scheme theoretic definition of the Gale transform for finite Gorenstein schemes. Eisenbud and Popescu [15] showed that certain finite determinantal subschemes of projective spaces defined by maximal minors of adjoint matrices of homogeneous polynomials of degree 1 are related by Veronese embeddings and a Gale transform. The result is technical to state. In [6] S. Diaz (Syracuse University) and I extended this result to adjoint matrices of multihomogeneous multilinear forms. Our subschemes lie in products of projective spaces and the Veronese embeddings are replaced with Segre embeddings. We are currently investigating the situations involving minors which are not maximal.

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