

# ALGEBRAIC INTERPRETATION OF A THEOREM OF CLEMENTS AND LINDSTRÖM

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ABSTRACT. We study Hilbert functions of quotients of the truncated polynomial ring  $k[x_1, \dots, x_n]/(x_1^{e_1+1}, x_2^{e_2+1}, \dots, x_n^{e_n+1})$ , where  $e_1 \geq e_2 \geq \dots \geq e_n \geq 1$  are integers. We use the work of Clements-Lindström to recover the well-known Macaulay's Theorem.

## 1. INTRODUCTION

Let  $R = k[x_1, \dots, x_n]$ , where  $k$  is a field, the  $x_i$  are indeterminates of degree 1, and  $I$  is a homogeneous ideal in  $R$ . Let  $S = R/I$ . Then  $S = \bigoplus_{i \geq 0} S_i$  is a graded ring. The Hilbert function of  $S$  is defined by  $H_S(i) = \dim_k S_i$ ,  $i \geq 0$ . By convention we take  $H_S(i) = 0$  if  $i < 0$ . A sequence  $\{c_i\}_{i \geq 0}$  such that  $c_i = H_S(i)$ ,  $i \geq 0$  for some such  $S$  is called an O-sequence. In particular we have  $c_0 = 1$ . It is convenient to take  $c_i = 0$  for  $i < 0$ . Macaulay characterized O-sequences combinatorially. Macaulay's characterization has been formulated by Stanley [7, Theorem 2.2 (i)  $\Leftrightarrow$  (iii)] in the form  $c_0 = 1$ ,  $c_i \geq 0$  for all  $i \geq 0$ , and  $c_{i+1} \leq c_i^{<i>}$  for  $i \geq 1$ , where  $c_i^{<i>}$  is defined in terms of binomial expansions. It is well known to commutative algebraists that the paper [1] of Clements and Lindström generalizes Macaulay's characterization of O-sequences to Hilbert functions of quotients of truncated polynomial rings of the form  $k[x_1, \dots, x_n]/(x_1^{e_1+1}, x_2^{e_2+1}, \dots, x_n^{e_n+1})$ , where  $e_1 \geq e_2 \geq \dots \geq e_n \geq 1$  are integers. However [1] is written in a combinatorial language and it seems not to be as well understood how to interpret [1] algebraically. Greene and Kleitman give an exposition of the work of Clements and Lindström in [3] (also in a primarily combinatorial language). The purpose of this expository note is to describe our present understanding of how things work algebraically. In section 2 we recall the results of Macaulay (as presented in [7]).

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In section 3 we interpret [1] in terms of rev-lex-segments and order ideals, and in section 4 we count the number of elements in rev-lex-segments, obtaining binomial expansions given in [3] that are similar to those used by Stanley in [7] to describe O-sequences. In section 5 (after Example 5.1) we describe through examples an algorithm to use these binomial expansions to work with the analogue of O-sequences for truncated polynomial rings. Finally in section 6 we indicate how Macaulay's characterization of O-sequences follows from section 5.

## 2. MACAULAY'S THEOREM

As above let  $R = k[x_1, \dots, x_n]$ .

**Definition 2.1.** [7, page 59] An *order ideal* of  $R$  is a non-empty set  $M$  of monomials in  $x_1, \dots, x_n$  such that if  $x \in M$  and  $y$  is a monomial dividing  $x$  then  $y \in M$ .

Note that if  $M$  is an order ideal then  $1 \in M$ . One might also refer to the set of exponents of an order ideal of  $R$  as an order ideal in  $\mathbb{N}^n$  ( $\mathbb{N} = \{0, 1, 2, \dots\}$ ). Explicitly an order ideal in  $\mathbb{N}^n$  is a non-empty subset  $\Lambda$  of  $\mathbb{N}^n$  such that if  $\alpha \in \Lambda$  and  $\beta <_{pr} \alpha$  then  $\beta \in \Lambda$  (where  $\beta <_{pr} \alpha$  means that  $\alpha - \beta$  has all coordinates  $\geq 0$ , with at least one coordinate  $> 0$ ).

Let  $\mathcal{M}$  be the set of all monomials in  $x_1, \dots, x_n$ . Then  $M$  is an order ideal of  $R$  if and only if  $\mathcal{M} \setminus M$  is the set of monomials in a monomial ideal  $I_M$  of  $R$ . If  $I$  is any homogeneous ideal of  $R$  then  $R/I$  has a  $k$ -basis which is (the canonical image of) an order ideal of  $R$  ([7, Theorem 2.1]). Therefore there is a monomial ideal  $J$  of  $R$  such that  $R/J$  has the same Hilbert function as  $R/I$ .

Of special interest in Macaulay's Theorem is the rev-lex order ideal. First recall the definition of deg-rev-lex ordering on  $\mathcal{M}$ , as given in recent expositions, such as [2, Chapter 2, Definition 6 (p.56)] (there called Graded Reverse Lex Order, or grevlex). Let  $x^\alpha = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$  and  $x^\beta = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$  where  $\alpha = (a_1, a_2, \dots, a_n)$  and  $\beta = (b_1, b_2, \dots, b_n)$ . Then  $x^\alpha > x^\beta$  means that either  $\sum a_i > \sum b_i$  (i.e.  $\deg x^\alpha > \deg x^\beta$ ) or  $\sum a_i = \sum b_i$  and the last non-zero coordinate of  $\alpha - \beta$  is negative. In this situation we will also say that  $\deg \alpha = \sum a_i$ ,  $\deg \beta = \sum b_i$  and  $\alpha > \beta$ , thereby putting a corresponding deg-rev-lex order on  $\mathbb{N}^n$  as well. In the following  $>$  will always denote deg-rev-lex order. When comparing monomials of the same degree we will often simply write "rev-lex" order. Some consequences of the definition are:

$$(a) \quad x_1 > x_2 > \cdots > x_n;$$

- (b) Any monomial containing only  $x_1, \dots, x_{i-1}$  is larger than any monomial of the same degree that contains  $x_i$  (where “contains” means that the variable in question occurs with a positive exponent), or more generally  $xx_i^a > yx_i^b$  if  $x$  and  $y$  are monomials in  $x_1, \dots, x_{i-1}$ ,  $xx_i^a$  and  $yx_i^b$  have the same degree, and  $a < b$ .

A *rev-lex segment of degree  $d$*  is a collection  $\mathcal{S}$  of monomials of degree  $d$  such that if  $y \in \mathcal{S}$  and  $y_1$  is a monomial of degree  $d$  such that  $y_1 > y$  then  $y_1 \in \mathcal{S}$ . That is  $\mathcal{S}$  consists of the largest  $|\mathcal{S}|$  elements of degree  $d$  in rev-lex order ( $|\ast|$  denoting cardinality). For example  $\{x_1^2\}$ ,  $\{x_1^2, x_1x_2\}$ ,  $\{x_1^2, x_1x_2, x_2^2\}$ ,  $\{x_1^2, x_1x_2, x_2^2, x_1x_3\}$  are all rev-lex segments, but  $\{x_1^2, x_1x_2, x_1x_3\}$  is not since  $x_2^2 > x_1x_3$  and  $x_2^2 \notin \{x_1^2, x_1x_2, x_1x_3\}$ .

**Definition 2.2.** A *rev-lex segment order ideal  $M$*  is an order ideal such that for all  $d$  the monomials  $M_d \subset M$  of degree  $d$  are a rev-lex segment.

Macaulay’s Theorem can be stated in the following manner.

**Theorem 2.3.** *There is a one-to-one correspondence between  $O$ -sequences (i.e. Hilbert functions of quotients  $k[x_1, \dots, x_n]/I$ ,  $I$  a homogeneous ideal) and rev-lex segment order ideals, defined as follows: to the sequence  $1, c_1, c_2, \dots, c_i, \dots$ , ( $c_1 \leq n$ ) define a set  $M$  by letting the degree  $d$  part  $M_d$  be the largest  $c_d$  monomials of degree  $d$  in  $x_1, \dots, x_n$ .*

The only thing that one needs to establish in order to prove Theorem 2.3 is that the set  $M$  constructed in the statement of the Theorem is in fact an order ideal. The following definition will be needed:

**Definition 2.4.** Let  $M$  be any set of monomials in  $R$ , and let  $M_d$  be the set of elements of  $M$  in degree  $d$ . Then the *compression  $C_\infty M$*  of  $M$  is the set of monomials consisting of the largest  $|M_d|$  elements of degree  $d$ , for all  $d \geq 0$ .

Given a bit of translation, [1, Corollary 2] says that the compression of an order ideal is an order ideal. This, together with Stanley’s observation [7, Theorem 2.1] that every Hilbert function comes from some order ideal, is exactly what we need to show that the set constructed in the statement of Theorem 2.3 is an order ideal. More details of the translation, will be described in the next sections.

### 3. TRUNCATED POLYNOMIAL RINGS.

Suppose we have a sequence of integers  $\underline{e} = \{e_1, \dots, e_n\}$  where  $e_1 \geq e_2 \geq e_3 \geq \dots \geq e_n \geq 1$ . Let  $R_{\underline{e}} = k[x_1, \dots, x_n]/(x_1^{e_1+1}, \dots, x_n^{e_n+1})$ . Let  $\mathcal{M}_{\underline{e}}$  be the set of monomials of  $R_{\underline{e}}$ . These are monomials  $x^\alpha$  with  $\alpha \in$

$\mathbb{N}_e^n = \{(a_1, a_2, \dots, a_n) \mid 0 \leq a_i \leq e_i \forall i\}$ . If also  $\beta = (b_1, b_2, \dots, b_n) \in \mathbb{N}_e^n$  then we say that  $x^\beta$  divides  $x^\alpha$  if  $b_i \leq a_i$ ,  $1 \leq i \leq n$ . If  $x^\beta$  divides  $x^\alpha$  we will define  $x^\alpha/x^\beta$  to be  $x^{\alpha-\beta}$ . We can regard  $\mathcal{M}_e$  as a subset of the monomials  $\mathcal{M}$  in  $R = k[x_1, \dots, x_n]$ . The definitions of order ideal and deg-rev-lex order can be adapted to  $\mathcal{M}_e$  in a natural way.

**Definition 3.1.** An *order ideal* of  $\mathcal{M}_e$  (or  $R_e$ ) is a non-empty set  $M \subseteq \mathcal{M}_e$  such that if  $x \in M$  and  $y$  is a monomial dividing  $x$  then  $y \in M$ . The exponent set  $\{\alpha \mid x^\alpha \in M\}$  of  $M$  will also be referred to as an order ideal (of  $\mathbb{N}_e^n$ ).

**Definition 3.2.** Let  $x^\alpha, x^\beta \in \mathcal{M}_e$ , where  $\alpha = (a_1, a_2, \dots, a_n)$  and  $\beta = (b_1, b_2, \dots, b_n)$  with  $a_i \leq e_i, b_i \leq e_i$  for all  $i$ . Then  $x^\alpha > x^\beta$  (in deg-rev-lex order of  $\mathcal{M}_e$ ) means that either  $\sum \alpha_i > \sum \beta_i$  (i.e.  $\deg x^\alpha > \deg x^\beta$ ) or  $\sum \alpha_i = \sum \beta_i$  and the last non-zero coordinate of  $\alpha - \beta$  is negative.

**Definition 3.3.** Let  $(\mathcal{M}_e)_d$  be the set of all monomials of degree  $d$  in  $\mathcal{M}_e$ . A *rev-lex segment of degree  $d$  in  $\mathcal{M}_e$*  is a subset  $\mathcal{S}$  of  $(\mathcal{M}_e)_d$  such that if  $M \in \mathcal{S}$  and  $M_1 \in (\mathcal{M}_e)_d$  with  $M_1 > M$  then  $M_1 \in \mathcal{S}$ . A *rev-lex segment order ideal  $M$  of  $\mathcal{M}_e$*  is an order ideal of  $\mathcal{M}_e$  such that for all  $d$  the monomials  $M_d \subset M$  of degree  $d$  are a rev-lex segment of degree  $d$ . A *final rev-lex segment of degree  $d$  in  $\mathcal{M}_e$*  is a subset  $\mathcal{S}'$  of  $(\mathcal{M}_e)_d$  such that if  $M \in \mathcal{S}'$  and  $M_1 \in (\mathcal{M}_e)_d$  with  $M_1 < M$  then  $M_1 \in \mathcal{S}'$ . The complement in  $(\mathcal{M}_e)_d$  of a rev-lex-segment is a final rev-lex-segment and vice versa. (Perhaps rev-lex segments should be called initial rev-lex segments, but we skip the ‘‘initial’’ as we use them more frequently than final rev-lex segments.)

Note that a subset  $M \subseteq \mathcal{M}_e$  is an order ideal of  $R_e$  if and only if  $M$  is an order ideal of  $R$ . Also, if  $x^\alpha, x^\beta \in \mathcal{M}_e$  then  $x^\alpha > x^\beta$  in  $\mathcal{M}_e$  if and only if  $x^\alpha > x^\beta$  in  $\mathcal{M}$ . But a rev-lex-segment in  $\mathcal{M}_e$  need not be a rev-lex-segment in  $\mathcal{M}$ . As in the case of  $R$  and  $\mathcal{M}$ , the complement  $N = \mathcal{M}_e \setminus M$  of an order ideal  $M \subseteq \mathcal{M}_e$  is the  $k$ -basis of a monomial ideal  $I_M$  of  $R_e$ .

As in the case of  $R$  we have

**Theorem 3.4.** *Let  $I$  be a homogeneous ideal in  $R_e$ . Then there is an order ideal  $M$  in  $\mathcal{M}_e$  whose canonical image in  $R_e/I$  forms a  $k$ -basis of  $R_e/I$ .*

*Proof.* The proof is the same as that of [7, Theorem 2.1]. (However we cannot just apply [7, Theorem 2.1] to the quotient of  $R$  with the same Hilbert function as  $R_e/I$  because an order ideal of  $\mathcal{M}$  need not be an order ideal of  $\mathcal{M}_e$ ).  $\square$

Note that  $R_{\underline{e}}/I_M$  has the same Hilbert function as  $R_{\underline{e}}/I$ .

**Definition 3.5.** Let  $M$  be any set of monomials in  $\mathcal{M}_{\underline{e}}$ , and let  $M_d$  be the set of elements of  $M$  in degree  $d$ . Then the *compression*  $CM$  of  $M$  (in  $\mathcal{M}_{\underline{e}}$ ) is the set of monomials consisting of the largest  $|M_d|$  elements in  $\mathcal{M}_{\underline{e}}$  of degree  $d$ , for all  $d \geq 0$ .

In order to apply the work of Clements and Lindström [1] we introduce some auxiliary notation.

**Definition 3.6.** (1) Let  $M$  be any set of monomials in  $\mathcal{M}_{\underline{e}}$ . Then the *last compression*  $LM$  of  $M$  (in  $\mathcal{M}_{\underline{e}}$ ) is the set of monomials consisting of the smallest  $|M_d|$  elements in  $\mathcal{M}_{\underline{e}}$  of degree  $d$ , for all  $d \geq 0$ .

(2) If  $m \in (\mathcal{M}_{\underline{e}})_d$  then  $\Gamma(m)$  is the set of all degree  $d-1$  monomial factors of  $m$ . If  $M$  is any subset of  $\mathcal{M}_{\underline{e}}$  then we define  $\Gamma(M) := \cup_{m \in M} \Gamma(m)$ .

(3) A set  $M \subseteq \mathcal{M}_{\underline{e}}$  is closed if  $\Gamma(M) \subseteq M$ .

(4) If  $m \in (\mathcal{M}_{\underline{e}})_d$  then  $P(m)$  is the set of all degree  $d+1$  monomial multiples of  $m$  in  $\mathcal{M}_{\underline{e}}$ . If  $M$  is any subset of  $\mathcal{M}_{\underline{e}}$  then we define  $P(M) := \cup_{m \in M} P(m)$ .

The operations  $C, \Gamma, L$ , and  $P$  are always relative to a chosen sequence  $\underline{e}$ . For simplicity of notation we omit indicating this explicitly. Sometimes we may wish to apply these operations to the exponent vector of a monomial in  $\mathcal{M}_{\underline{e}}$ . For example if  $0 \leq a_i \leq e_i$  for  $1 \leq i \leq n$  then  $\Gamma((a_1, a_2, \dots, a_n)) = \cup_{i=1}^n \{(a_1, a_2, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n)\}$ , where if  $a_i = 0$  then that element of the union is omitted. The action of  $P$  on exponent vectors is similar, replacing  $a_i - 1$  by  $a_i + 1$  and omitting  $(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_n)$  if  $a_i = e_i$ .

**Remark 3.7.** We note that  $M$  is an order ideal if and only if  $\Gamma(M) \subseteq M$  (i.e.  $M$  is closed), and that  $M$  is the complement of an order ideal if and only if  $P(M) \subseteq M$ .

The results that we need from [1] are the following.

**Theorem 3.8.** [1, Theorem] *Let  $M \subseteq \mathcal{M}_{\underline{e}}$ . Then  $\Gamma(CM_d) \subseteq C(\Gamma(M_d))$  for each  $d \geq 1$ .*

**Corollary 3.9.** [1, Corollary 1] *Let  $M$  be as above. Then  $P(LM_d) \subseteq L(P(M_d))$  for each  $d \geq 0$ .*

**Corollary 3.10.** [1, Corollary 2] *Let  $M$  be as above. If  $M$  is closed then  $CM$  is closed, i.e. the compression of an order ideal is a rev-lex-segment order ideal.*

Clements and Lindström work only with vectors  $(a_1, a_2, \dots, a_n)$  where  $0 \leq a_i \leq k_i$  for a given set of integers  $1 \leq k_1 \leq k_2 \leq \dots \leq k_n$ . We regard these as exponent vectors of monomials. They work with lexicographic order, and their compression is the first (i.e. smallest) vectors in lexicographic order. However the following easy lemma shows that reversing the order of the coordinates and replacing  $<$  by  $>$  turns their definitions of  $\Gamma, C, L, P$  into the ones given above, so we can apply their results verbatim. Their increasing sequence  $1 \leq k_1 \leq k_2 \leq \dots \leq k_n$  is turned into our decreasing sequence  $e_1 \geq e_2 \geq e_3 \geq \dots \geq e_n \geq 1$  because of the reversal of order of the coordinates. Let  $<_{\text{lex}}$  denote lexicographic order. As always,  $<$  denotes reverse lexicographic order.

**Lemma 3.11.** *If  $\alpha \in \mathbb{N}^n$  denote by  $\alpha^r$  the vector obtained by reversing the order of the coordinates. Suppose that  $\deg(\alpha) = \deg(\beta)$ . Then  $\alpha > \beta$  if and only if  $\alpha^r <_{\text{lex}} \beta^r$ .*

*Proof.* If  $\deg(\alpha) = \deg(\beta)$  then  $\alpha > \beta$  if and only if the last non-zero coordinate of  $\alpha - \beta$  is negative if and only if the last non-zero coordinate of  $\beta - \alpha$  is positive if and only if the first non-zero coordinate of  $\beta^r - \alpha^r$  is positive if and only if  $\alpha^r <_{\text{lex}} \beta^r$ .  $\square$

Putting Theorem 3.4, Remark 3.7, and Corollary 3.10 together we have

**Theorem 3.12.** *There is a one-to-one correspondence between Hilbert functions  $H = \{c_i\}_{i \geq 0}$  of quotients of  $R_{\underline{e}}$  by a homogeneous ideal, and rev-lex-segment order ideals of  $\mathcal{M}_{\underline{e}}$ . The order ideal corresponding to  $H$  is obtained by taking the largest  $c_i$  monomials of  $\mathcal{M}_{\underline{e}}$  in degree  $i$  for all  $i \geq 0$ .*

From Theorem 3.8 and Corollary 3.9 we will obtain growth conditions on the Hilbert function of a quotient of  $R_{\underline{e}}$ .

**Definition 3.13.** If  $x^\alpha$  is a monomial of degree  $d$  in  $\mathcal{M}_{\underline{e}}$  define  $\mathcal{L}(x^\alpha)$  to be the set of all monomials of degree  $d$  in  $\mathcal{M}_{\underline{e}}$  that are greater than or equal to  $x^\alpha$ .

Clearly every rev-lex-segment  $\mathcal{S}$  of degree  $d$  in  $\mathcal{M}_{\underline{e}}$  equals  $\mathcal{L}(x^\alpha)$ , where  $x^\alpha$  is the smallest element of  $\mathcal{S}$ .

Now we prove a couple of lemmas that are crucial to our way of looking at things.

**Lemma 3.14.** *Let  $\mathcal{S} = \mathcal{L}(x^\alpha)$  be a rev-lex-segment of degree  $d > 0$  in  $\mathcal{M}_{\underline{e}}$ , where  $\alpha = (a_1, \dots, a_n)$ . Suppose that  $r$  is the smallest index such that  $a_r > 0$ . Then the set  $\mathcal{T} = \Gamma(\mathcal{S})$  is equal to  $\mathcal{L}(x^\alpha/x_r)$ .*

*Proof.* We have  $\mathcal{S} = C(\mathcal{S})$  so by Theorem 3.8 (with  $M_d = \mathcal{S}$ ) we have  $\mathcal{T} \subseteq C\mathcal{T}$ . But  $|\mathcal{T}| = |C\mathcal{T}|$ , so  $\mathcal{T} = C\mathcal{T}$ . We have  $\alpha = (0, \dots, 0, a_r, a_{r+1}, \dots, a_n)$  and let  $\beta = (0, \dots, 0, a_r - 1, a_{r+1}, \dots, a_n)$  so that  $x^\alpha/x_r = x^\beta$ . Clearly  $x^\alpha/x_r \in \mathcal{T}$  so it suffices to prove that if  $x^{\alpha'} \in (\mathcal{M}_e)_{d-1}$  and  $x^{\alpha'} < x^\beta$  then  $x^{\alpha'} \notin \mathcal{T}$ . Let  $s$  be the largest index in which the coordinates of  $\alpha'$  and  $\beta$  differ (necessarily with that of  $\alpha'$  being larger). Since  $\beta$  is 0 in coordinates 1 through  $r - 1$  and  $\deg \beta = \deg \alpha' = d - 1$  we must have  $s > r$ . Then  $x_i x^{\alpha'} < x^\alpha$  for any  $i$  such that  $x_i x^{\alpha'} \in \mathcal{M}_e$  because the coordinates of the exponent vector of  $x_i x^{\alpha'}$  in the range  $s$  through  $n$  can only be the same or larger than those of  $x^{\alpha'}$  and at least the  $s$  coordinate of the latter is already greater than  $a_s$ .  $\square$

**Lemma 3.15.** *Let  $\mathcal{S} = \mathcal{L}(x^\alpha)$  be a rev-lex-segment of degree  $d$  in  $\mathcal{M}_e$ , where  $\alpha = (a_1, a_2, \dots, a_n)$ . Let  $\mathcal{U}$  (or  $\mathcal{U}(\mathcal{S})$  if it is necessary to specify the starting rev-lex-segment) be the set of all degree  $d + 1$  monomials in  $\mathcal{M}_e$  all of whose degree  $d$  factors are in  $\mathcal{S}$ . Suppose that  $r$  is the smallest index such that  $a_r > 0$  and let  $\beta = (0, \dots, 0, b_r, b_{r+1}, \dots, b_n)$  where  $b_r = a_r + 1$  and  $b_i = a_i$  if  $r + 1 \leq i \leq n$ . If  $x^{\beta'} \in (\mathcal{M}_e)_{d+1}$  then  $x^{\beta'} \in \mathcal{U}$  if and only if  $\beta' \geq \beta$  in the deg-rev-lex order of  $\mathbb{N}^n$ . If  $a_r < e_r$  then  $x^\beta = x_r x^\alpha \in (\mathcal{M}_e)_{d+1}$  and  $\mathcal{U} = \mathcal{L}(x_r x^\alpha)$ . If  $a_r = e_r$  then  $\mathcal{U} = \mathcal{L}(x^\gamma)$  where  $x^\gamma$  is the smallest monomial of  $\mathcal{M}_e$  that is larger than  $x^\beta$ .*

*Proof.* We have  $(\mathcal{M}_e)_{d+1} \setminus \mathcal{U} = P((\mathcal{M}_e)_d \setminus \mathcal{S})$  and  $\mathcal{S}' := (\mathcal{M}_e)_d \setminus \mathcal{S}$  is a final rev-lex-segment. Therefore  $L(\mathcal{S}') = \mathcal{S}'$  and by Corollary 3.9 with  $M_d = \mathcal{S}'$  we have  $P(\mathcal{S}') \subseteq LP(\mathcal{S}')$ . Since  $|P(\mathcal{S}')| = |LP(\mathcal{S}')|$  we have  $P(\mathcal{S}') = LP(\mathcal{S}')$ . Therefore  $P(\mathcal{S}')$  is a final rev-lex-segment, and hence  $\mathcal{U}$  is a rev-lex-segment.

If  $x^\beta \in (\mathcal{M}_e)_{d+1}$  (equivalently  $e_r > a_r$ ) then clearly every degree  $d$  monomial factor of  $x^\beta$  is greater than or equal to  $x^\alpha$  so  $x^\beta \in \mathcal{U}$ . If  $x^{\beta'} \in (\mathcal{M}_e)_{d+1}$  and if  $x^{\beta'} > x^\beta$  then  $x^{\beta'} \in \mathcal{U}$  since  $\mathcal{U}$  is a rev-lex-segment. If  $x^\beta \notin (\mathcal{M}_e)_{d+1}$  (equivalently  $e_r = a_r$ ) then we can define  $\underline{f} = (f_1, f_2, \dots, f_n)$  by  $f_i = e_i + 1, 1 \leq i \leq n$ , so that  $\mathcal{M}_e \subset \mathcal{M}_{\underline{f}}$  and  $x^\beta \in (\mathcal{M}_{\underline{f}})_{d+1}$ . Let  $\mathcal{U}_{\underline{f}}$  be the set of monomials in  $(\mathcal{M}_{\underline{f}})_{d+1}$  all of whose degree  $d$  factors are greater than or equal to  $x^\alpha$ . If  $x^{\beta'} \in (\mathcal{M}_e)_{d+1}$  and if  $x^{\beta'} > x^\beta$  then  $x^{\beta'} \in \mathcal{U}_{\underline{f}}$  since  $\mathcal{U}_{\underline{f}}$  is a rev-lex-segment in  $(\mathcal{M}_{\underline{f}})_{d+1}$ . But clearly  $\mathcal{M}_e \cap \mathcal{U}_{\underline{f}} = \mathcal{U}$  so again  $x^{\beta'} \in \mathcal{U}$ .

Now assume that  $\deg \beta' = d + 1$  and  $\beta' < \beta$ . Then  $\beta' = (\dots, b'_r, \dots, b'_s, b'_{s+1}, \dots, b'_n)$  where  $b'_s > b_s$  and  $b_i = b'_i$  for  $s + 1 \leq i \leq n$ . We have  $r < s \leq n$ , because otherwise the degree of  $\beta'$  would be greater than  $d + 1$ . If  $b'_j \neq 0$  for some  $j < s$  then  $x^{\beta'}/x_j < x^\alpha$ , i.e.  $x^{\beta'}/x_j \notin \mathcal{S}$  so

$x^{\beta'} \notin \mathcal{U}$ . If  $b'_s > b_s + 1$  then  $x^{\beta'}/x_s < x^\alpha$  so again  $x^{\beta'} \notin \mathcal{U}$ . We can't have both  $b'_j = 0$  for  $j < s$  and  $b'_s = b_s + 1$  because  $b_r \geq 2$ . Hence always  $x^{\beta'} \notin \mathcal{U}$ .

The final assertion of the Lemma is obvious.  $\square$

If  $a_r = e_r$  in Lemma 3.15, then  $\gamma$  can be described as follows. If  $r > 1$  then  $\gamma = (0, \dots, 0, 1, e_r, a_{r+1}, \dots, a_n)$ . If  $r = 1$  pick  $j$ ,  $2 < j \leq n$  to be the smallest integer such that  $a_j > 0$  and  $\sum_{i=2}^{j-1} (e_i - a_i) \geq 2$ . Then  $\gamma = (0, \dots, 0, a, e_s, \dots, e_{j-1}, a_j - 1, a_{j+1}, \dots, a_n)$  where  $a$  and  $s$  are chosen so that  $\gamma$  is of degree  $d + 1$ . The summation condition guarantees that such a  $\gamma$  exists. (We need the sum  $\geq 2$  so that each of  $e_1 + 1$  and  $a_j$  can be decreased by 1 without forcing some other coordinate to be greater than the corresponding value  $e_i$ .) If no such  $j$  exists, then  $\mathcal{U}$  is empty.

The  $n$ -tuple  $\gamma$  is perhaps best described by an example. Suppose  $\underline{e} = (10, 10, 10, 10, 10, 10)$  and let  $\alpha = (10, 9, 10, 0, 0, 1)$  (so that  $r = 1$ ). Then  $\beta = (11, 9, 10, 0, 0, 1)$  and  $j = 6$  yielding  $\gamma = (0, 1, 10, 10, 10, 0)$ .

**Theorem 3.16.** *Let  $R_{\underline{e}}/I$  be a graded quotient of  $R_{\underline{e}}$  with Hilbert function  $H$ . Suppose that  $H(d) = c$ . Let  $\mathcal{S}$  be the rev-lex-segment of degree  $d$  consisting of the  $c$  largest monomials of degree  $d$  in  $\mathcal{M}_{\underline{e}}$ . Then  $H(d - 1)$  must be at least the cardinality of the rev-lex-segment  $\mathcal{T}$  of degree  $d - 1$  consisting of all degree  $d - 1$  factors of elements of  $\mathcal{S}$ .*

*Proof.* By Theorem 3.4 there is an order ideal  $M$  of  $\mathcal{M}_{\underline{e}}$  which forms a  $k$ -basis of  $R_{\underline{e}}/I$ . We have  $\mathcal{S} = CM_d$  and  $\mathcal{T} = \Gamma(\mathcal{S})$ . By Theorem 3.8,  $\Gamma(CM_d) \subseteq C(\Gamma(M_d))$  so  $H(d - 1) = |M_{d-1}| \geq |\Gamma(M_d)| = |C(\Gamma(M_d))| \geq |\Gamma(\mathcal{S})| = |\mathcal{T}|$  as claimed.  $\square$

**Theorem 3.17.** *Let  $R_{\underline{e}}/I$  be a graded quotient of  $R_{\underline{e}}$  with Hilbert function  $H$ . Suppose that  $H(d) = c$ . Let  $\mathcal{S}$  be the rev-lex-segment of degree  $d$  consisting of the  $c$  largest monomials of degree  $d$  in  $\mathcal{M}_{\underline{e}}$ . Then  $H(d + 1)$  can be at most the cardinality of the rev-lex-segment  $\mathcal{U}$  of degree  $d + 1$  consisting of those degree  $d + 1$  monomials in  $\mathcal{M}_{\underline{e}}$  all of whose degree  $d$  factors are in  $\mathcal{S}$ .*

*Proof.* By Theorem 3.4 there is an order ideal  $M$  of  $\mathcal{M}_{\underline{e}}$  which forms a  $k$ -basis of  $R_{\underline{e}}/I$ . Let  $N_i = (\mathcal{M}_{\underline{e}})_i \setminus M_i$  for all  $i$ . By Corollary 3.9 we have  $P(LN_d) \subseteq L(P(N_d))$ . Therefore  $(\mathcal{M}_{\underline{e}})_{d+1} \setminus \mathcal{U} = P((\mathcal{M}_{\underline{e}})_d \setminus \mathcal{S}) = P(LN_d) \subseteq L(P(N_d))$ . The latter has the same cardinality as  $P(N_d) \subseteq N_{d+1}$ . Therefore  $|\mathcal{U}| \geq |M_{d+1}| = H(d + 1)$  as claimed.  $\square$

**Remark 3.18.** The condition  $e_1 \geq e_2 \geq \dots \geq e_n$  is necessary to apply [1] (as explained in the discussion after Corollary 3.10). The following example shows that without this condition the theory does not work

at all. Let  $e_1 = 1$  and  $e_2 = 3$ . Then  $R_{\underline{e}} = k[x_1, x_2]/(x_1^2, x_2^4)$ , which has Hilbert function  $1, 2, 2, 2, 1, 0 \rightarrow$ . The graded quotient ring  $R_{\underline{e}}/(x_1x_2)$  has Hilbert function  $1, 2, 1, 1, 0 \rightarrow$ . But if we take the rev-lex segments of degrees  $0, 1, 2, 3$  containing respectively the largest  $1, 2, 1, 1$  elements we get  $\{1, x_1, x_2, x_1x_2, x_1x_2^2\}$ , which is not an order ideal because  $x_1x_2^2$  has factor  $x_2^2$  which is not in the set. But if we take  $e_1 = 3, e_2 = 1$  we get  $\{1, x_1, x_2, x_1^2, x_1^3\}$ , which is an order ideal.

**Remark 3.19.** Hilbert functions exhibiting the extreme behaviour in Theorems 3.16 and 3.17 in fact do exist. Suppose  $c > 0$  is the proposed value of a Hilbert function in degree  $d$ . Let  $\mathcal{S}$  be the rev-lex-segment of degree  $d$  consisting of the  $c$  largest monomials of degree  $d$  in  $\mathcal{M}_{\underline{e}}$ . Let  $M$  be the order ideal of all factors of elements of  $\mathcal{S}$  together with the set  $\mathcal{U}$  described in Lemma 3.15. Then  $R_{\underline{e}}/I_M$  has Hilbert function  $H$  with  $H(d) = c$ ,  $H(d-1)$  the value given by Theorem 3.16 and  $H(d+1)$  the value given by Theorem 3.17.

#### 4. BINOMIAL EXPANSIONS

In this section  $\underline{e}$ ,  $R_{\underline{e}}$  and  $\mathcal{M}_{\underline{e}}$  are as in the previous section. We will count the number of elements in the various rev-lex-segments discussed in Section 3, leading to binomial expansions similar to that used by Stanley in describing  $h^{<i>}$  in [7].

Now we count the number of elements in a deg-rev-lex segment. First a preliminary definition.

**Definition 4.1.** Let

$$\binom{e_1, \dots, e_r}{d}$$

denote the number of monomials of degree  $d$  in  $x_1, \dots, x_r$  in which the exponent of  $x_i$  does not exceed  $e_i$ . If  $d = 0$  define the value to be 1.

We will refer to the  $\binom{e_1, \dots, e_r}{d}$  as “binomials”, but note that  $\binom{e_1}{d}$  is not the usual binomial coefficient. It equals 1 if  $0 \leq d \leq e_1$  and 0 if  $d > e_1$ . We will refer to  $d$  as the “denominator” of the binomial and  $e_1, \dots, e_r$  as the “numerator”. There does not seem to be a simple explicit expression for  $\binom{e_1, e_2, \dots, e_r}{d}$ . However there is a generating function, namely  $\binom{e_1, e_2, \dots, e_r}{d}$  is the coefficient of  $t^d$  in  $\prod_{i=1}^r (\sum_{j=0}^{e_i} x^j)$ .

**Lemma 4.2.** Let  $x^\alpha \in \mathcal{M}_{\underline{e}}$  be a monomial of degree  $d$  in the  $x_i$  ( $1 \leq i \leq n$ ), where  $\alpha = (a_1, \dots, a_s, 0, \dots, 0)$ , with  $a_s > 0$  for some  $s$  satisfying  $1 \leq s \leq n$ . Then the number of elements in  $\mathcal{L}(x^\alpha)$  is given by the following recursively defined binomial expansion.

(a)

$$\binom{e_1, \dots, e_s}{d}$$

if  $x^\alpha$  is the smallest monomial in  $(\mathcal{M}_{\underline{e}'})_d$ , ( $\underline{e}' = (e_1, \dots, e_s)$ ).

(b) otherwise

$$\sum_{i=0}^{a_s-1} \binom{e_1, \dots, e_{s-1}}{d-i} + S$$

where  $S$  is the expansion for  $\mathcal{L}(x^\beta)$  where  $\beta = (a_1, \dots, a_{s-1}, 0, \dots, 0)$ .

**Definition 4.3.** The expression given by Lemma 4.2 will be referred to as the *degree  $d$   $\underline{e}$ -binomial expansion of  $\mathcal{L}(x^\alpha)$*  (or of the integer  $|\mathcal{L}(x^\alpha)|$ ).

**Remark 4.4.** If  $s = 1$  we must have case (a) so the recursion terminates. If  $s > 1$  then we have case (a) if and only if  $\alpha = (0, \dots, 0, b_i, e_{i+1}, \dots, e_s, 0, \dots, 0)$  for some  $i$ ,  $1 \leq i \leq s$  with  $b_i + \sum_{j=i+1}^s e_j = d$ . If  $e_s \geq d$  then we have case (a) if and only if  $x^\alpha = x_s^d$ .

**Remark 4.5.** The summation in (b) enumerates respectively the number of monomials in  $\mathcal{L}(x^\alpha)$  with no  $x_s$ , with  $x_s$  to the first power, the second power, respectively, up to exponent  $a_s - 1$ . The  $S$  term enumerates those monomials with factor  $x_s^{a_s}$ . We could also define the recursion in (b) as  $\binom{e_1, \dots, e_{s-1}}{d} + S_1$  where  $S_1$  is the binomial expansion enumerating  $\mathcal{L}(x^{\alpha'})$  where  $\alpha' = (a_1, \dots, a_s - 1, 0, \dots, 0)$ .

**Remark 4.6.** If we are given the cardinality  $a = |\mathcal{L}(x^\alpha)|$  of the revlex-segment instead of its smallest element  $x^\alpha$  we can obtain the same binomial expansion as follows. The sequence  $\left\{ \binom{e_1}{d}, \binom{e_1, e_2}{d}, \binom{e_1, e_2, e_3}{d}, \dots \right\}$  is strictly increasing once it becomes non-zero. If the  $n$ -th term is still 0 there are no monomials of degree  $d$  in  $x_1, \dots, x_n$  and  $\mathcal{L}(x^\alpha)$  is empty. Otherwise take the first term  $\binom{e_1, \dots, e_s}{d}$  in the expansion to be the largest of  $\left\{ \binom{e_1}{d}, \binom{e_1, e_2}{d}, \binom{e_1, e_2, e_3}{d}, \dots, \binom{e_1, e_2, \dots, e_n}{d} \right\}$  that is less than or equal to  $a$ . If  $a = \binom{e_1, \dots, e_s}{d}$  then we have case (a) of Lemma 4.2 and this is the entire binomial expansion. Otherwise repeat the process with  $a - \binom{e_1, \dots, e_s}{d}$  using  $\left\{ \binom{e_1}{d-1}, \binom{e_1, e_2}{d-1}, \binom{e_1, e_2, e_3}{d-1}, \dots, \binom{e_1, e_2, \dots, e_n}{d-1} \right\}$ . We are done when the total of our expansion is  $a$ . (If the first non-zero value in  $\left\{ \binom{e_1}{d}, \binom{e_1, e_2}{d}, \binom{e_1, e_2, e_3}{d}, \dots, \binom{e_1, e_2, \dots, e_n}{d} \right\}$  is greater than  $a$  then, by Lemma 4.2 (b), take the first term in the expansion to be the last coefficient  $\binom{e_1, \dots, e_s}{d}$  that has value 0).

**Remark 4.7.** Let  $a = |\mathcal{L}(x^\alpha)|$  where  $\alpha = (a_1, \dots, a_p, 0, \dots, 0)$  where  $p$  is the largest value of  $i$  such that  $a_i > 0$ . Then the degree  $d$   $\underline{e}$ -binomial expansion of  $a$  has the following form. If  $x^\alpha$  is the smallest monomial of degree  $d$  in  $x_1, \dots, x_p$  then there is one (which we think of as  $1 = a_{p+1} + 1$ ) term  $\binom{e_1, e_2, \dots, e_p}{d}$ . Otherwise there are  $a_p$  terms

$$\sum_{i=0}^{a_p-1} \binom{e_1, e_2, \dots, e_{p-1}}{d-i}$$

then  $a_{p-1}$  terms

$$\sum_{i=a_p}^{a_{p-1}+a_p-1} \binom{e_1, e_2, \dots, e_{p-2}}{d-i}$$

and so on, until the expansion terminates with case (a) of Lemma 4.2, in which case there is one extra term with the same numerator. The denominators are  $d, d-1, \dots$ , successively decreasing by one, with the final denominator at least one. Each numerator  $e_1, \dots, e_{i-1}$  except the last occurs  $a_i \leq e_i$  times. The final numerator  $e_1, \dots, e_s$  also occurs  $\leq e_{s+1}$  times because if  $a_{s+1} = e_{s+1}$  we would have terminated earlier with case (a). Thus we obtain the expansion of [3, page 70]. The form of the expansion is perhaps best understood with examples, which we give in the next section.

**Remark 4.8.** If in Lemma 4.2 some of the coordinates of  $\alpha$  are greater than the corresponding  $e_i$  then the expansion described in the Lemma still gives the number of monomials in the rev-lex-segment consisting of all monomials in  $\mathcal{M}_{\underline{e}}$  of degree  $d$  whose exponent vector is greater than  $\alpha$  in the deg-rev-lex ordering of  $\mathbb{N}^n$ . Coordinates that are too large are taken into account automatically by the definition of the  $\binom{e_1, \dots, e_r}{d}$ .

**Definition 4.9.** Let  $a, d \geq 1$ . Then  $a_{\underline{e}}^{<d>}$  is the integer obtained by increasing by 1 all the denominators in the degree  $d$   $\underline{e}$ -binomial expansion of  $a$ , and  $a_{\underline{e}}^{\leq d}$  is the integer obtained by decreasing by 1 all the denominators in the degree  $d$   $\underline{e}$ -binomial expansion of  $a$ .

**Theorem 4.10.** Let  $\mathcal{S} = \mathcal{L}(x^\alpha)$  be a rev-lex-segment of degree  $d$  as in Lemma 3.14. Suppose that the binomial expansion for  $c = |\mathcal{L}(x^\alpha)|$  given by Lemma 4.2 is

$$\binom{e_1, \dots, e_l}{d} + \dots + \binom{e_1, \dots, e_s}{d-a}$$

(where  $l = p$  or  $l = p - 1$  with  $p$  as in Remark 4.7). Then the rev-lex-segment  $\mathcal{T}$  (in degree  $d - 1$ ) of Lemma 3.14 contains

$$c_{<d>}^e = \binom{e_1, \dots, e_l}{d-1} + \dots + \binom{e_1, \dots, e_s}{d-a-1}$$

elements.

*Proof.* This follows from the definition of the expansion in Lemma 4.2 and the description of  $\mathcal{T}$  in Lemma 3.14. Each term in the expansion of  $\mathcal{L}(x^\alpha/x_r)$  enumerates monomials of degree one lower than the corresponding term for  $\mathcal{L}(x^\alpha)$  (involving the the same variables).  $\square$

**Corollary 4.11.** *Suppose that the Hilbert function of a graded quotient of  $R_{\underline{e}}$  has value  $c$  in degree  $d$ . Then the Hilbert function in degree  $d - 1$  must have value at least  $c_{<d>}^e$ .*

*Proof.* This follows from Theorem 4.10 and Theorem 3.16.  $\square$

**Theorem 4.12.** *Let  $\mathcal{S} = \mathcal{L}(x^\alpha)$  be a rev-lex-segment of degree  $d$  as in Lemma 3.15. Suppose that the binomial expansion for  $c = \mathcal{L}(x^\alpha)$  given by Lemma 4.2 is*

$$\binom{e_1, \dots, e_l}{d} + \dots + \binom{e_1, \dots, e_s}{d-a}$$

(where  $l = p$  or  $l = p - 1$  with  $p$  as in Remark 4.7). Then the rev-lex-segment  $\mathcal{U}$  (in degree  $d + 1$ ) of Lemma 3.15 contains

$$c_{\underline{e}}^{<d>} = \binom{e_1, \dots, e_l}{d+1} + \dots + \binom{e_1, \dots, e_s}{d-a+1}$$

elements.

*Proof.* This follows from the definition of the expansion in Lemma 4.2 and the description of  $\mathcal{U}$  in Lemma 3.15 (together with Remark 4.8). Each term in the expansion of  $\mathcal{L}(x_r x^\alpha)$  enumerates monomials of degree one higher than the corresponding term for  $\mathcal{L}(x^\alpha)$  (involving the same variables).  $\square$

**Corollary 4.13.** *Suppose that the Hilbert function of a graded quotient of  $R_{\underline{e}}$  has value  $c$  in degree  $d$ . Then the Hilbert function in degree  $d + 1$  can have value at most  $c_{\underline{e}}^{<d>}$ .*

*Proof.* This follows from Theorem 4.12 and Theorem 3.17.  $\square$

B. Richert and S. Sabourin [6] have obtained similar results which we think are essentially equivalent to Corollary 4.13. They also obtained the algorithm mentioned before Example 5.2. We believe that they used a different method of proof. In addition, J. Mermin has obtained

results concerning compressed ideals [4] and lexlike sequences [5] which are of interest.

## 5. EXAMPLES AND AN ALGORITHM

**Example 5.1.** Let  $\underline{e}$  be  $\{4, 3, 3, 2\}$  and  $x^\alpha = x_2^3 x_3^2 x_4^2$  (so  $\alpha = (0, 3, 2, 2)$ ). Then  $x^\alpha$  is not the smallest monomial of degree 7 in  $x_1, x_2, x_3, x_4$  so we are in case (b) of Theorem 4.2. The expansion starts out  $\binom{4,3,3}{7}$  enumerating monomials of degree 7 in  $x_1, x_2, x_3$  followed by  $\binom{4,3,3}{6}$  enumerating monomials of degree 6 in  $x_1, x_2, x_3$  (which are to be multiplied by  $x_4$ ). Now we find the expansion of  $\mathcal{L}(x_2^3 x_3^2)$ . Again we are in case (b), so we get 2 more terms  $\binom{4,3}{5}$  enumerating monomials of degree 5 in  $x_1$  and  $x_2$  (which are to be multiplied by  $x_4^2$ ) and  $\binom{4,3}{4}$  enumerating monomials of degree 4 in  $x_1$  and  $x_2$  (which are to be multiplied by  $x_3 x_4^2$ ). We conclude with the expansion of  $\mathcal{L}(x_2^3)$ . This is the smallest monomial of degree 3 in  $x_1$  and  $x_2$  so we are in case (a) of Theorem 4.2, yielding that the degree 7  $\underline{e}$ -binomial expansion of  $x_2^3 x_3^2 x_4^2$  is

$$\binom{4,3,3}{7} + \binom{4,3,3}{6} + \binom{4,3}{5} + \binom{4,3}{4} + \binom{4,3}{3} = 10 + 13 + 3 + 4 + 4.$$

Note that indeed we have  $a_4 = 2$  terms with numerator 4 3 3, and  $a_3 + 1 = 3$  terms with numerator 4 3 (the extra term coming from the termination with case (a)) as observed in Remark 4.8. The total is 34, so  $x_2^3 x_3^2 x_4^2$  is the 34-th degree 7 monomial in deg-rev-lex order, when powers are restricted to  $x_1^4, x_2^3, x_3^3, x_4^2$ .

Furthermore we have

$$\begin{aligned} 34_{\underline{e}}^{<7>} &= \binom{4,3,3}{8} + \binom{4,3,3}{7} + \binom{4,3}{6} + \binom{4,3}{5} + \binom{4,3}{4} \\ &= 6 + 10 + 2 + 3 + 4 \\ &= 25 \end{aligned}$$

and

$$\begin{aligned} 34_{<\underline{e}7>}^e &= \binom{4,3,3}{6} + \binom{4,3,3}{5} + \binom{4,3}{4} + \binom{4,3}{3} + \binom{4,3}{2} \\ &= 13 + 14 + 4 + 4 + 3 \\ &= 38. \end{aligned}$$

Therefore if a graded quotient of  $k[x_1, x_2, x_3, x_4]/(x_1^5, x_2^4, x_3^4, x_4^3)$  has Hilbert function with value 34 in degree 7 then the Hilbert function has value at most 25 in degree 8 and at least value 38 in degree 6.

Using the notation introduced in Lemmas 3.14 and 3.15, we observe that  $\mathcal{S} = \mathcal{L}(x_2^3 x_3^2 x_4^2)$ ,  $\mathcal{T} = \mathcal{L}(x_2^2 x_3^2 x_4^2)$  and  $\mathcal{U}$  can be enumerated as

if formally  $\mathcal{U} = \mathcal{L}(x_2^4 x_3^2 x_4^2)$ . The terms of the binomial expansion of  $\mathcal{L}(x_2^2 x_3^2 x_4^2)$  enumerate respectively monomials of degree 6 in  $x_1, x_2, x_3$ , degree 5 in  $x_1, x_2, x_3$ , degree 4 in  $x_1, x_2$ , degree 3 in  $x_1, x_2$ , and degree 2 in  $x_1, x_2$  (the last enumerating  $\mathcal{L}(x_2^2)$ ), all of degree 1 lower than the corresponding term of expansion of  $\mathcal{L}(x_2^3 x_3^2 x_4^2)$  illustrating the proof of Theorem 4.10. Similarly the terms in the binomial expansion of  $\mathcal{L}(x_2^4 x_3^2 x_4^2)$  are of degree one higher than the corresponding terms in the expansion of  $\mathcal{L}(x_2^3 x_3^2 x_4^2)$ . Actually  $\mathcal{U} = \mathcal{L}(x_1 x_2^3 x_3^2 x_4^2)$ , the enumeration of which gives the same expansion.

Note that the sequence  $\binom{e_1, \dots, e_r}{i}$ ,  $i \geq 0$  is the Hilbert function of  $R_{\underline{e}}$  or equivalently of a complete intersection  $CI(e_1 + 1, e_2 + 1, \dots, e_r + 1)$  defined by a regular sequence consisting of  $r$  homogeneous polynomials of degrees  $e_1 + 1, \dots, e_r + 1$  in  $k[x_1, \dots, x_r]$ . This, together with the construction in Remark 4.6 starting from  $a = |\mathcal{L}(x^\alpha)|$  suggests the following algorithm for finding the degree  $d$   $\underline{e}$ -binomial expansion of  $a$ . We illustrate it with Example 5.1 above, in which  $\underline{e} = \{4, 3, 3, 2\}$ . Make a table with rows  $CI(5)$ ,  $CI(5, 4)$ ,  $CI(5, 4, 4)$ ,  $CI(5, 4, 4, 3)$ , the columns from left to right corresponding to degrees  $0, 1, 2, \dots$

$CI(5)$	1	1	1	1	0	0	0	0	0	0	0	0
$CI(5, 4)$	1	2	3	<u>4</u>	<u>4</u>	<u>3</u>	2	1	0	0	0	0
$CI(5, 4, 4)$	1	3	<b>6</b>	<b>10</b>	13	14	<u>13</u>	<u>10</u>	6	3	1	0
$CI(5, 4, 4, 3)$	1	4	10	19	29	37	40	37	29	19	10	4

In finding the 7-binomial expansion of 34 we form the sequence  $\binom{4}{7}, \binom{4,3}{7}, \binom{4,3,3}{7}, \binom{4,3,3,2}{7}$  and select the largest which is less than or equal to 34. This is the degree 7 column 0 1 10 37, so start with  $10 = \binom{4,3,3}{7}$ . Now subtract  $34 - 10 = 24$  and go down the degree 6 column, 0 2 13 40, selecting  $13 = \binom{4,3,3}{6}$ , which is the largest number in the degree six column which is less than or equal to 24. Now subtract  $24 - 13 = 11$  and go down the degree 5 column selecting  $3 = \binom{4,3}{5}$ . Subtract  $11 - 3 = 8$  and go down the degree 4 column selecting  $4 = \binom{4,3}{4}$ , subtract yielding  $8 - 4 = 4$ , then the degree 3 column selecting  $4 = \binom{4,3}{3}$  which completes the expansion. These entries in the table are underlined. To find  $34_{\underline{e}}^{<7>}$  shift each number one unit to the right and add, yielding  $34_{\underline{e}}^{<7>} = 6 + 10 + 2 + 3 + 4 = 25$  as obtained in Example 5.1. To find  $34_{\underline{e}}^{>7}$  shift each number one unit to the left and add, obtaining  $34_{\underline{e}}^{>7} = 13 + 14 + 4 + 4 + 3 = 38$ , also as was obtained in Example 5.1. Now we work out a couple of other examples, using the table above.

**Example 5.2.** Let  $\underline{e} = \{4, 3, 3, 2\}$  as above, and  $x^\alpha = x_1 x_4^2$ . The degree 3  $\underline{e}$ -binomial expansion of  $\mathcal{L}(x^\alpha)$  is  $\binom{4,3,3}{3} + \binom{4,3,3}{2} + \binom{4}{1} = 10 + 6 + 1$  (the 10 from the degree 3 column (counting all monomials of degree 3

in  $x_1, x_2, x_3$ ), the 6 from the degree 2 column (counting all monomials of degree 2 in  $x_1, x_2, x_3$  which are to be multiplied by  $x_4$ ), and the 1 from the degree 1 column counting  $x_1x_4^2$  itself). These entries are in bold in the table. This gives  $17 = 10 + 6 + 1$ . Shifting left in the table gives  $17_{<3>}^e = 6 + 3 + 1 = \binom{4,3,3}{2} + \binom{4,3,3}{1} + \binom{4}{0}$ . Here  $\mathcal{T} = \mathcal{L}(x_4^2)$  and  $\mathcal{L}(x_4^2)$  has been enumerated as 6 monomials of degree 2 in  $x_1, x_2, x_3$ , three monomials of degree 1 in  $x_1, x_2, x_3$  (which are to be multiplied by  $x_4$ ) and one monomial 1 of degree 0 (which is to be multiplied by  $x_4^2$ ). This is not the “official” Lemma 4.2 expansion of  $\mathcal{L}(x_4^2)$  which would be  $\binom{4,3,3,2}{2} = 10$ , but it is correct enumeration. If we shift right in the table we obtain  $17_{<3>}^e = 13 + 10 + 1 = \binom{4,3,3}{4} + \binom{4,3,3}{3} + \binom{4}{2}$ . The 13 enumerates all monomials of degree 4 in  $x_1, x_2, x_3$  of which there are only 13 because the exponent of  $x_2$  and  $x_3$  is at most 3, the 10 enumerates all monomials of degree 3 in  $x_1, x_2, x_3$  (which are to be multiplied by  $x_4$ ) and the 1 enumerates  $x_1^2x_4^2$  itself (1 monomial of degree 2 in  $x_1$ , which is multiplied by  $x_4^2$ ). Here  $\mathcal{U} = \mathcal{L}(x_1^2x_4^2)$ .

**Example 5.3.** Let  $\underline{e} = \{4, 3, 3, 2\}$  as above, and  $x^\alpha = x_1^4x_2^2x_4$ . This example illustrates that we need not have  $\Gamma(\mathcal{U}(\mathcal{L}(x^\alpha))) = \mathcal{L}(x^\alpha)$ . The degree 7  $\underline{e}$ -binomial expansion of  $\mathcal{S} = \mathcal{L}(x^\alpha)$  is  $\binom{4,3,3}{7} + \binom{4}{6} + \binom{4}{5} + \binom{4}{4} = 10 + 0 + 0 + 1 = 11$  (the 10 from the degree 7 column (counting all monomials of degree 7 in  $x_1, x_2, x_3$ ), the first 0 from the degree 6 column (counting all monomials of degree 6 in  $x_1$ , of which there are none because  $e_1 = 4$ , but if there were any they would be multiplied by  $x_4$ ), the second 0 counting all monomials of degree 5 in  $x_1$  (again there are none, but if there were any they would be multiplied by  $x_2x_4$ ), and the 1 counting all monomials of degree 4 in  $x_1$ , of which there is one, namely  $x_1^4$  which when multiplied by  $x_2^2x_4$  yields  $x^\alpha$ ). Shifting left in the table gives  $11_{<7>}^e = 13 + 0 + 1 + 1 = \binom{4,3,3}{6} + \binom{4}{5} + \binom{4}{4} + \binom{4}{3}$ . Here  $\mathcal{T} = \mathcal{L}(x_1^3x_2^2x_4)$  and  $\mathcal{L}(x_1^3x_2^2x_4)$  has been enumerated as 13 monomials of degree 6 in  $x_1, x_2, x_3$ , zero monomials of degree 5 in  $x_1$  (which would be multiplied by  $x_4$  if there were any), one monomial  $x_1^4$  of degree 4 in  $x_1$  (which is to be multiplied by  $x_2x_4$ ) and one monomial  $x_1^3$  of degree 3 in  $x_1$  (which is to be multiplied by  $x_2^2x_4$  yielding  $x_1^3x_2^2x_4$  itself). Note that we have to keep  $\binom{4}{6}$  and  $\binom{4}{5}$  in the degree 7 binomial expansion of 11 even though they are 0 in order to correctly compute  $11_{<7>}^e$  by lowering denominators (actually keeping  $\binom{4}{5}$  suffices here, but it seems preferable to keep  $\binom{4}{6}$  as well). If we shift right in the table we obtain  $11_{<7>}^e = 6 + 0 + 0 + 0 = \binom{4,3,3}{8} + \binom{4}{7} + \binom{4}{6} + \binom{4}{5} = 6$ . This counts all monomials of degree 8 in  $x_1, x_2, x_3$ , the smallest of which is  $x_1^2x_2^3x_3^3$  so here  $\mathcal{U} = \mathcal{L}(x_1^2x_2^3x_3^3)$ . As noted in the proof of

Lemma 3.15 and Remark 4.8 this expansion of  $11_e^{<7>}$  is obtained by enumerating the formal expression  $\mathcal{L}(x_1^5 x_2^2 x_4)$  in the usual way. The three 0's in the resulting expansion of  $11_e^{<7>}$  reflect the fact that there are no monomials of degree 7 in  $\mathcal{M}_e$  involving only  $x_1$  and  $x_2$  which are formally greater than or equal to  $x_1^5 x_2^2$ . In fact the next monomial of degree 8 in  $\mathcal{M}_e$  smaller than  $x_1^2 x_2^3 x_3^3$  is  $x_1^4 x_2^3 x_4$  and this contains a degree 7 factor  $x_1^3 x_2^3 x_4$  that is not in  $\mathcal{L}(x^\alpha)$  so  $\mathcal{U}$  cannot be any larger than  $\mathcal{L}(x_1^2 x_2^3 x_3^3)$ . The official expansion of  $\mathcal{L}(x_1^2 x_2^3 x_3^3)$  is just the single term  $\binom{4,3,3}{8}$  and we must use it if we go back down using Theorem 4.10, obtaining  $\Gamma(\mathcal{U}(\mathcal{L}(x_1^4 x_2^2 x_4))) = \Gamma(\mathcal{L}(x_1^2 x_2^3 x_3^3)) = \mathcal{L}(x_1 x_2^3 x_3^3)$ .

## 6. RECOVERING MACAULAY'S GROWTH CONDITIONS

If we make the  $e_i$  (which we will informally write  $\infty$ ) larger than the degree under discussion we recover the usual form of Macaulay's Theorem. Thus  $\binom{\infty, \dots, \infty}{d}$  (with  $n$   $\infty$ 's in the numerator) means the number of monomials of degree  $d$  in  $n$  variables (without restriction on exponents) which is  $C(d+n-1, d)$ , where we write the usual binomial coefficient  $\frac{a!}{b!(a-b)!}$  as  $C(a, b)$  to avoid conflict with the notation in Definition 4.1. The expansion of Lemma 4.2 becomes the usual binomial expansion mentioned in [7, page 60]. Corollaries 4.11 and 4.13 then become

**Theorem 6.1.** *Let  $R$  be a graded algebra over a field  $k$ , generated by elements of degree 1. Suppose that the Hilbert function of  $R$  has value  $h$  in degree  $i$ . Suppose that  $h$  has binomial expansion*

$$h = C(n_i, i) + C(n_{i-1}, i-1) + \cdots + C(n_j, j)$$

where  $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$ . Define

$$h^{<i>} = C(n_i + 1, i + 1) + C(n_{i-1} + 1, i) + \cdots + C(n_j + 1, j + 1)$$

and

$$h_{<i>} = C(n_i - 1, i - 1) + C(n_{i-1} - 1, i - 2) + \cdots + C(n_j - 1, j - 1).$$

Then

- (a) *The Hilbert function of  $R$  in degree  $i+1$  has value at most  $h^{<i>}$ .*
- (b) *The Hilbert function of  $R$  in degree  $i-1$  has value at least  $h_{<i>}$ .*

Furthermore the two extreme cases are realized by the quotient of a polynomial ring by a lex-segment ideal.

Part (a) of the Theorem is of course very well known and is stated explicitly in [7, Theorem 2.2(c)]. Part (b) is less often stated.

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