Cal Poly Department of Mathematics

Puzzle of the Week

Nov 11 - ~~17~~ Dec 1, 2016

An unit mass object in the x, y-plane is at rest at the origin at time $t = 0$. It is acted on by two forces at all times: first a force of gravity pulls it down (negative y-axis direction) with a constant acceleration of magnitude $g > 0$; second a force with magnitude proportional to the object's speed, with proportionality constant $c > 0$, pushes in a direction that is at a 90^o counter-clockwise rotation from its direction of motion.

Where is the object at time $t = \frac{3\pi}{c}$? Can you describe the type of curve its trajectory sketches?

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 329

before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in the next email announcement. Anybody associated to Cal Poly is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The object will be at position $x = 3\pi g/c^2$, $y = -2g/c^2$ at time $t = 3\pi/c$. The path traveled is that of an inverted cycloid.

Specifically, let $\vec{r}$ denote its position vector, and let $\vec{v}$, $\vec{a}$ denote its velocity and acceleration respectively. Then the conditions in the problem give the equation

$$\vec{a} = -g\vec{j} + cR\vec{v}, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In coordinates this gives the pair of ODEs:

$$\begin{aligned} \ddot{x} &= -c\dot{y} \\ \ddot{y} &= -g + c\dot{x} \end{aligned}$$

Solving these and applying the initial conditions leads to the solution

$$x(t) = \frac{g}{c^2}(ct - \sin(ct))$$
$$y(t) = \frac{g}{c^2}(\cos(ct) - 1)$$

which gives the solution above.

Alternatively one can solve the first order vector equation

$$\frac{d}{dt}\vec{v} = -g\vec{j} + cR\vec{v}$$

If $M = e^{-cRt}$ then $\frac{d}{dt}M = -cRM = -cMR$, so that the above vector ODE can be rewritten as

$$\frac{d}{dt}(M\vec{v}) = -gM\vec{j}$$

which leads to

$$\vec{v} = -gM^{-1}\int M\vec{j}\,dt$$

or, plugging in for M, evaluating the integral, and using $\vec{v}(0) = \vec{0}$ and $R^{-1}\vec{j} = \vec{i}$, we get

$$\vec{v} = \frac{g}{c}(I - e^{cRt})\vec{i}$$

Integrating again and using $\vec{r}(0) = \vec{0}$ we find

$$\vec{r} = \frac{g}{c^2}\left(ct\vec{i} + (e^{cRt} - I)\vec{j}\right)$$

A quick calculation reveals that $e^{cRt} = \cos(ct)I + \sin(ct)R$. Applying this, and simplifying we recover our above solution, but in vector form:

$$\vec{r}(t) = \frac{g}{c^2}\left((ct - \sin(ct))\vec{i} + (\cos(ct) - 1)\vec{j}\right)$$