Consider the ellipse \( x^2/a^2 + y^2/b^2 = 1 \). It is well known that there is no “simple” formula for length \( L \) of its perimeter. However one might approximate the length with either of the formulas:

\[
L \approx \pi(a + b), \quad \text{or} \quad L \approx 2\pi\sqrt{ab}
\]

Both of these approximations are exact when \( b = a \). In this problem we investigate which is a better approximation when \( b \) is very close to (but not equal to) \( a \).

To do this we choose units so that \( a = 1 \), and let \( L \) denote the length of the perimeter of the ellipse \( x^2 + y^2/b^2 = 1 \). Let \( b = 1 + \epsilon \) and calculate

\[
L - \pi(1 + b) \quad \text{and} \quad L - 2\pi\sqrt{b}
\]

correct to second order in \( \epsilon \). Use this to determine which is the better approximation, and by what approximate factor it is better.

Solution: If you were stumped by this one don’t feel too bad, I borrowed this problem from the 1950 Putnam Exam — I found it too interesting to resist making it into a Puzzle of the Week.

The first approximation is about three times as good as the second approximation.

To see this we first compute \( L \) to second order in \( \epsilon \). Parametrize the ellipse by \( x = \cos \theta \), and
$y = b \sin \theta$, for $0 \leq \theta \leq 2\pi$. Then using the techniques we learn in Math 143 we can calculate:

$$L = \int_0^{2\pi} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} \, d\theta$$

$$= \int_0^{2\pi} \sqrt{\sin^2 \theta + (1 + \epsilon)^2 \cos^2 \theta} \, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + (2\epsilon + \epsilon^2) \cos^2 \theta} \, d\theta$$

$$= \int_0^{2\pi} \left( 1 + \frac{1}{2}(2\epsilon + \epsilon^2) \cos^2 \theta + \left( \frac{1}{2} \right)^2 (2\epsilon + \epsilon^2) \cos^4 \theta + \ldots \right) \, d\theta$$

$$= 2\pi + \frac{1}{2}(2\epsilon + \epsilon^2)\pi - \frac{1}{8}(2\epsilon + \epsilon^2)^2 \frac{3\pi}{4} + \ldots$$

$$= 2\pi + \pi\epsilon + \frac{\pi}{8}\epsilon^2 + \ldots$$

Now we also note that

$$\pi(1 + b) = 2\pi + \pi\epsilon$$

and

$$2\pi\sqrt{b} = 2\pi(1 + \epsilon)^{1/2} = 2\pi \left( 1 + \frac{1}{2}\epsilon + \left( \frac{1}{2} \right)^2 \epsilon^2 + \ldots \right) = 2\pi + \pi\epsilon - \frac{2\pi}{8}\epsilon^2 + \ldots$$

Putting everything together we find:

$$L - \pi(1 + b) = \frac{\pi}{8} \epsilon^2 + \mathcal{O}(\epsilon^3)$$

$$L - 2\pi\sqrt{b} = \frac{3\pi}{8} \epsilon^2 + \mathcal{O}(\epsilon^3)$$

So the first approximation is better, and we can see the error is about $1/3$ that of the error in the second approximation.