Cal Poly Department of Mathematics

Puzzle of the Week

May 19 - May 31, 2016

Consider the ellipse $x^2/a^2 + y^2/b^2 = 1$. It is well known that there is no "simple" formula for length L of its perimeter. However one might approximate the length with either of the formulas:

$$L \approx \pi(a+b), \quad \text{or} \quad L \approx 2\pi\sqrt{ab}$$

Both of these approximations are exact when $b = a$. In this problem we investigate which is a *better* approximation when b is very close to (but not equal to) a.

To do this we choose units so that $a = 1$, and let L denote the length of the perimeter of the ellipse $x^2 + y^2/b^2 = 1$. Let $b = 1 + \epsilon$ and calculate

$$L - \pi(1+b) \quad \text{and} \quad L - 2\pi\sqrt{b}$$

correct to second order in ϵ. Use this to determine which is the better approximation, and by what approximate factor it is better.

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 329

before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in the next email announcement. Anybody associated to Cal Poly is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> If you were stumped by this one don't feel too bad, I borrowed this problem from the 1950 Putnam Exam — I found it too interesting to resist making it into a Puzzle of the Week.

The first approximation is about three times as good as the second approximation.

To see this we first compute L to second order in ϵ. Parametrize the ellipse by $x = \cos\theta$, and

$y = b\sin\theta$, for $0 \le \theta \le 2\pi$. Then using the techniques we learn in Math 143 we can calculate:

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{\sin^2\theta + (1+\epsilon)^2\cos^2\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{1 + (2\epsilon+\epsilon^2)\cos^2\theta}\, d\theta \\
&= \int_0^{2\pi} \left(1 + \frac{1}{2}(2\epsilon+\epsilon^2)\cos^2\theta + \binom{1/2}{2}(2\epsilon+\epsilon)^2\cos^4\theta + \dots\right) d\theta \\
&= 2\pi + \frac{1}{2}(2\epsilon+\epsilon^2)\pi - \frac{1}{8}(2\epsilon+\epsilon^2)^2\frac{3\pi}{4} + \dots \\
&= 2\pi + \pi\epsilon + \frac{\pi}{8}\epsilon^2 + \dots
\end{aligned}$$

Now we also note that

$$\pi(1+b) = 2\pi + \pi\epsilon$$

and

$$2\pi\sqrt{b} = 2\pi(1+\epsilon)^{1/2} = 2\pi\left(1 + \frac{1}{2}\epsilon + \binom{1/2}{2}\epsilon^2 + \dots\right) = 2\pi + \pi\epsilon - \frac{2\pi}{8}\epsilon^2 + \dots$$

Putting everything together we find:

$$\begin{aligned}
L - \pi(1+b) &= \frac{\pi}{8}\epsilon^2 + \mathcal{O}(\epsilon^3) \\
L - 2\pi\sqrt{b} &= \frac{3\pi}{8}\epsilon^2 + \mathcal{O}(\epsilon^3)
\end{aligned}$$

So the first approximation is better, and we can see the error is about 1/3 that of the error in the second approximation.