Cal Poly Department of Mathematics

Puzzle of the Week

May 5 - May 18, 2016

Calculate the limit

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\frac{1}{\sqrt{n^2+3^2}}+\ldots+\frac{1}{\sqrt{n^2+n^2}}\right)$$

Solutions should be submitted to Morgan Sherman:

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before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in the next email announcement. Anybody associated to Cal Poly is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The limit evaluates to $\log(1+\sqrt{2})$.

The trick is to recognize the sum as a Riemann sum:

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\frac{1}{\sqrt{n^2+3^2}}+\ldots+\frac{1}{\sqrt{n^2+n^2}}\right)\\
&\quad=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+(1/n)^2}}+\frac{1}{\sqrt{1+(2/n)^2}}+\frac{1}{\sqrt{1+(3/n)^2}}+\ldots+\frac{1}{\sqrt{1+(n/n)^2}}\right)\\
&\quad=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1/n}{\sqrt{1+(1/k)^2}}=\int_0^1\frac{dx}{\sqrt{1+x^2}}=\log(x+\sqrt{x^2+1})\big|_0^1=\log(1+\sqrt{2})
\end{aligned}$$

Note that the function $f(x)=1/\sqrt{1+x^2}$ is continuous on $[0,1]$ and that the above sum is the Riemann sum of f on $[0,1]$ with n subintervals, each of equal width, and with right endpoints chosen.