Cal Poly Department of Mathematics

Puzzle of the Week

April 21 - May 5, 2016

Which is larger:

$$-1+\frac{2^1}{0!}+\frac{2^2}{1!}+\frac{2^3}{2!}+\frac{2^4}{3!}+\dots$$

or

$$\frac{1^2}{0!}+\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\dots$$

??

(Note: each series can be computed exactly...)

Solutions should be submitted to Morgan Sherman:

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before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in the next email announcement. Anybody associated to Cal Poly is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u>

Note that in general, if $f(x)=\sum_{n=0}^{\infty}a_nx^n$ then we have $\frac{d}{dx}(xf(x))=\sum_{n=0}^{\infty}(n+1)a_nx^n$. Hence we find that

$$(x+1)e^x=\frac{d}{dx}(xe^x)=\sum_{n=0}^{\infty}(n+1)\frac{x^n}{n!}$$

and that

$$(x^2+3x+1)e^x=\frac{d}{dx}((x+1)e^x)=\sum_{n=0}^{\infty}(n+1)^2\frac{x^n}{n!}$$

In particular, if $x=1$ then we find

$$5e=\frac{1^2}{0!}+\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\dots$$

So we know the value of the second series. For the first we calculate

$$-1+\frac{2^1}{0!}+\frac{2^2}{1!}+\frac{2^3}{2!}+\frac{2^4}{3!}+\ldots=-1+2\sum_{n=0}^{\infty}\frac{2^n}{n!}=-1+2e^2$$

These two numbers can be calculated explicitly and we find that $2e^2-1\approx 13.778$ and $5e\approx 13.591$, so the first series is larger.

Alternative we can write the difference of the first series minus the second as

$$-1+(\frac{2}{1}-\frac{1}{1})+(\frac{4}{1}-\frac{4}{1})+(\frac{8}{2}-\frac{9}{2})+(\frac{16}{6}-\frac{16}{6})+(\frac{32}{24}-\frac{25}{24})+(\frac{64}{120}-\frac{36}{120})+\sum_{n=7}^{\infty}\frac{2^n-n^2}{n!}$$

which simplifies to

$$\frac{1}{40}+\sum_{n=7}^{\infty}\frac{2^n-n^2}{n!}$$

and now one just needs to show that each term of the truncated series above is positive, which comes down to showing that $2^n>n^2$ for $n\geq 7$.