Cal Poly Department of Mathematics

Puzzle of the Week

Nov 19 - Dec 2, 2015

Consider the following construction: Begin with a unit circle and its circumscribed equilateral triangle; then circumscribe about the triangle a new circle, and circumscribe about it a square; continue in this way, so that after constructing a regular n-gon you circumscribe it with a circle and circumscribe the new circle with a regular $n+1$-gon. If $r_2 = 1$ denotes the radius of the first circle, and in general we let r_n denote the radius of the circle which circumscribed the regular n-gon, then what is a formula for r_n? Correct to four decimal places, what is r_{1000}? Do you think r_n tends to a finite value, or infinity, as $n \to \infty$? (No proof required)

Solutions should be submitted to Morgan Sherman:

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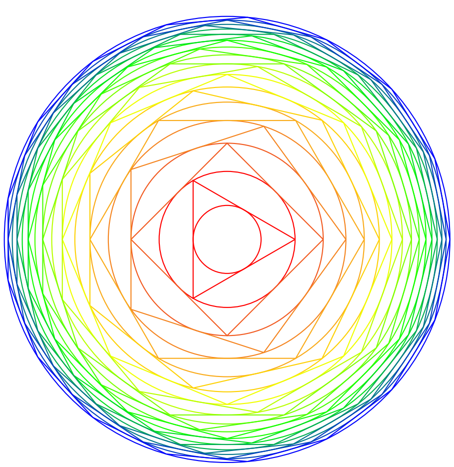
before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in the next email announcement. Anybody associated to Cal Poly is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution:

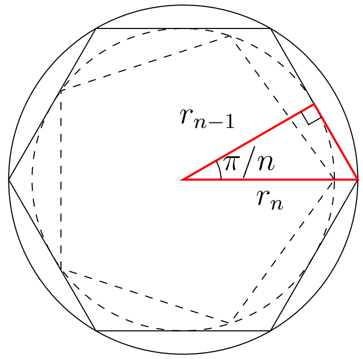


A formula for r_n is given by:

$$r_n = \prod_{k=3}^{n} \sec \frac{\pi}{k} = (\sec \frac{\pi}{3})(\sec \frac{\pi}{4}) \cdot \ldots \cdot (\sec \frac{\pi}{n})$$

from which we compute $r_{1000} \approx 8.6572$. In fact the sequence $\{r_n\}_{n=3}^{\infty}$ converges.

The formula for r_n follows from the below picture:



In general we have $r_n = r_{n-1} \sec \frac{\pi}{n}$, from which we get the formula given above.

To see the sequence converges first note that the terms r_n are all greater than 1 and they are monotonically increasing. So either the sequence $\{r_n\}$ converges or it diverges to infinity. Now note that

$$\log r_n = \log \sec \frac{\pi}{3} + \log \sec \frac{\pi}{4} + \log \sec \frac{\pi}{5} + \ldots + \log \sec \frac{\pi}{n}.$$

Hence the sequence $\{r_n\}$ converges if and only if the sum $\sum_{n=3}^{\infty} \log \sec \frac{\pi}{n}$ converges. We do this by a limit comparison test with the series $\sum \frac{1}{n^2}$ which we know converges:

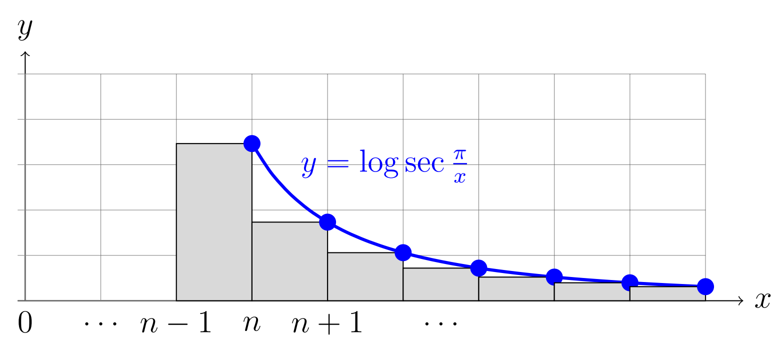
$$\begin{aligned} \lim_{x\to\infty} \frac{\log \sec \frac{\pi}{x}}{\frac{1}{x^2}} &= \lim_{x\to\infty} \frac{-\frac{\pi}{x^2} \tan \frac{\pi}{x}}{-\frac{2}{x^3}} \\ &= \lim_{x\to\infty} \frac{\pi^2}{2} \frac{\tan \frac{\pi}{x}}{\frac{\pi}{x}} = \frac{\pi^2}{2} \end{aligned}$$

The first equality is due to L'Hôpital's Rule. Hence the sum $\sum_{n=3}^{\infty} \log \sec \frac{\pi}{n}$ converges, and hence the sequence $\{r_n\}$ also converges.

To get a better idea of the limiting value we numerically compute

$$r_{1000} \approx 8.6572$$

(I used the computer algebra system *Sage*). To see how good an approximation this is we can use the integral test, as depicted below:



$\sum_{k=n}^{\infty} \log \sec \frac{\pi}{k} \le \log \sec \frac{\pi}{n} + \int_n^{\infty} \log \sec \frac{\pi}{x}\, dx$

First we obtain the more precice bounds

$$\frac{1}{2}u^2 \le \log\sec u \le \frac{1}{2}u^2 + \frac{1}{4}u^4$$

which holds, for example, for $0 \le u \le \pi/3$. (This can be obtained by integrating the corresponding bounds $\theta \le \tan\theta \le \theta + \theta^3$ which are easy to verify for positive $\theta < \pi/3$.) Then for any $n \ge 3$ we have

$$\begin{aligned}
E_n &:= \log\sec\frac{\pi}{n+1} + \log\sec\frac{\pi}{n+2} + \log\sec\frac{\pi}{n+3} + \dots \\
&\le \int_n^\infty \log\sec\frac{\pi}{x}\,dx \\
&= \pi\int_0^{\pi/n} \frac{1}{u^2}\log\sec u\,du \qquad (u = \frac{\pi}{x}) \\
&\le \pi\int_0^{\pi/n} (\frac{1}{2} + \frac{1}{4}u^2)\,du \\
&= \pi\left(\frac{\pi}{2n} + \frac{\pi^3}{12n^3}\right)
\end{aligned}$$

We also compute (again using the integral test)

$$\begin{aligned}
E_n &\ge \int_{n+1}^\infty \log\sec\frac{\pi}{x}\,dx \\
&= \pi\int_0^{\pi/(n+1)} \frac{1}{u^2}\log\sec u\,du \\
&\ge \pi\int_0^{\pi/(n+1)} \frac{1}{2}\,du = \frac{\pi^2}{2n+2}
\end{aligned}$$

Hence for every n we find

$$\frac{\pi^2}{2n+2} \le E_n \le \frac{\pi^2}{2n} + \frac{\pi^4}{12n^3}.$$

Note that for any n we also have

$$r := \lim_{k\to\infty} r_k = r_n \cdot \exp(E_n).$$

Using the above bounds for E_n for $n = 1000$ we find that $8.6996\cdot\exp(\frac{\pi^2}{2002}) \le r \le 8.6996\cdot\exp(\frac{\pi^2}{2000} + \frac{\pi^4}{1.2\times 10^{10}})$ which comes to

$$8.700015 \le r \le 8.700058$$

In fact the value of r is known to be approximately $8.700036625\dots$. It is known as the *polygon circumscribing constant*; you can find more information from the On-Line Encyclopedia of Integer Sequences where the digits of this constant are recorded in entry A051762.