Cal Poly Department of Mathematics

Puzzle of the Week

Nov 5 - 18, 2015

Fix a constant $a > 1$ and evaluate, with proof, the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{na^n}$$

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 329

before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in the next email announcement. Anybody associated to Cal Poly is welcome to make a submission.

http://www.calpoly.edu/˜sherman1/puzzleoftheweek

<u>*Solution:*</u> The sum converges to $\log \frac{a}{a-1}$.

First note that $\int_a^\infty \frac{1}{x^{n+1}}\,dx = \frac{1}{na^n}$. Hence we calculate:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{na^n} &= \sum_{n=1}^{\infty} \int_a^\infty \frac{1}{x^{n+1}}\,dx \\
&= \int_a^\infty \sum_{n=1}^{\infty} \frac{1}{x^{n+1}}\,dx \\
&= \int_a^\infty \frac{\frac{1}{x^2}}{1-\frac{1}{x}}\,dx \\
&= \int_a^\infty \frac{1}{x(x-1)}\,dx \\
&= \int_a^\infty \left(-\frac{1}{x} + \frac{1}{x-1}\right)dx \\
&= \log \frac{x-1}{x}\bigg|_a^\infty = \log \frac{a}{a-1}
\end{aligned}$$