Cal Poly Department of Mathematics

Puzzle of the Week

Oct 22-Nov 4, 2015

From Tom O'Neil:

Find all *rational* triples (a, b, c) for which a, b, and c are exactly all the roots of

$$x^3 + ax^2 + bx + c = 0.$$

Solutions should be submitted to Morgan Sherman:

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before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in the next email announcement. Anybody associated to Cal Poly is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>Solution:</u> There are three such triples: $(0, 0, 0), (1, -2, 0)$, and $(1, -1, -1)$.

The assumption in the question can be expressed by writing $x^3 + ax^2 + bx + c = (x - a)(x - b)(x - c)$ which leads to the three equations

$$-a = a + b + c, \quad b = ab + bc + ca, \quad -c = abc.$$

From the third equation we find that either $c = 0$ or $ab = -1$.

If $c = 0$ we find from the first equation that $b = -2a$, and then from the second equation that $a = a^2$. So either $a = 0$ or $a = 1$. The former gives the point $(0, 0, 0)$ while the latter gives the point $(1, -2, 0)$.

If $c \neq 0$ then we must have $ab = -1$. In this case we find, with the help of the first equation, that $b = -\frac{1}{a}$ and $c = \frac{1}{a} - 2a$. Plugging these into the second equation and clearing denominators we find

$$2a^4 - 2a^2 - a + 1 = 0$$

Now we use the Rational Roots Theorem to note that any rational root must be one of $\pm 1, \pm \frac{1}{2}$, and of these only $a = 1$ works, giving the last point $(1, -1, -1)$.