Cal Poly Department of Mathematics

Puzzle of the Week

May 21 - June 3, 2015

From Tom O'Neil:

Let C be the circle of radius R, centered at $(R, 0)$ in the plane. Let C' be the circle of radius $r < 2R$, centered at the origin. Let Y denote the point where C' meets the positive y-axis, and let Z denote the point where C' and C intersect in the first quadrant. The line passing through Y and Z meets the positive x-axis at the point X. What happens to X as $r \to 0^+$?

Solutions should be submitted to Morgan Sherman:

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before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The point X tends to the point $(4R, 0)$.

The point Y has coordinates $(0, r)$, while the point Z has coordinates (x, y) satisfying

$$x^2 + y^2 = r^2 \quad \text{and} \quad (x - R)^2 + y^2 = R^2.$$

Solving for x, y in the first quadrant and denoting their values by α, β, respectively, we find that

$$\alpha = \alpha(r) = \frac{r^2}{2R}, \quad \beta = \beta(r) = \sqrt{r^2 - \left(\frac{r^2}{2R}\right)^2} = r - \frac{r^3}{8R^2} + O(r^5)$$

(where we've used the binomial series $\sqrt{1+z} = 1 + \frac{1}{2}z + \binom{1/2}{2}z^2 + \ldots$).

So the line through Y and Z has slope $m = m(r) = \frac{\beta - r}{\alpha} = \frac{-\frac{r^3}{8R^2} + O(r^5)}{\frac{r^2}{2R}} = -\frac{r}{4R} + O(r^3)$, and the point X will have x-coordinate satisfying $0 - r = m(x - \alpha)$. Solving for x we find:

$$x = \frac{-r}{m} + \alpha = \frac{1}{\frac{1}{4R} + O(r^2)} + \frac{r^2}{2R} \longrightarrow 4R \quad \text{as } r \to 0^+.$$