Cal Poly Department of Mathematics

Puzzle of the Week

May 8 - May 20, 2015

From Tom O'Neil:

Find the minimum value of the quantity

$$\Big(r-1\Big)^2+\Big(\frac{s}{r}-1\Big)^2+\Big(\frac{t}{s}-1\Big)^2+\Big(\frac{4}{t}-1\Big)^2$$

where r, s, t are real numbers such that $1 \leq r \leq s \leq t \leq 4$.

Solutions should be submitted to Morgan Sherman:

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before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The minimum value is $4(\sqrt{2}-1)^2 = 12-8\sqrt{2}$.

If we set

$$x = r-1,\ y = \frac{s}{r}-1,\ z = \frac{t}{s}-1,\ w = \frac{4}{t}-1$$

then we see, after a bit of algebra, that it is equivalent to find the minimum value of

$$x^2+y^2+z^2+w^2, \quad \text{subject to } (x+1)(y+1)(z+1)(w+1) = 4.$$

This is solvable using Lagrange multipliers, for example. This leads to the system of equations

$$2x(x+1) = \lambda\gamma,\ 2y(y+1) = \lambda\gamma,\ 2z(z+1) = \lambda\gamma,\ 2w(w+1) = \lambda\gamma$$

where we have set $\gamma = (x+1)(y+1)(z+1)(w+1)$. This leads quickly to the condition $x = y = z = w$ if we assume each is nonnegative (which does not affect the final value). Plugging back in to the constraint we find

$$x = y = z = w = \sqrt{2}-1$$

which gives the solution above.