Cal Poly Department of Mathematics

Puzzle of the Week

April 23 - May 7, 2015

Fix a positive constant $c > 0$ and consider the parabola p given by $y = cx^2$. For any positive $t > 0$ let P_t be the point (t, ct^2) on the parabola, and let ℓ_t denote the line passing through P_t and normal to p at this point. Denote by $f(t)$ the x-coordinate of the other point of intersection of ℓ_t and p (which will lie in the second quadrant). Find the $t > 0$ which maximizes $f(t)$, or show that no such t exists.

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The function f is maximized when $t = 1/(\sqrt{2}c)$.

The line normal to p at P_t has slope $-1/(2ct)$, and therefore is given by

$$y - ct^2 = \frac{-1}{2ct}(x - t).$$

This will intersect p whenever we also have $y = cx^2$. Substituting in for y we find that x must satisfy

$$cx^2 - ct^2 = \frac{-1}{2ct}(x - t).$$

Since we are looking for the point of intersection where $x \neq t$ we can solve for x and and find

$$f(t) = x = \frac{-1}{2c^2t} - t.$$

Now, given that $t > 0$ we find:

$$f'(t) = \frac{1}{2c^2t^2} - 1 = 0 \iff t = \frac{1}{\sqrt{2}c}, \quad \text{and} \quad f''(t) = \frac{-1}{c^2t^3} < 0 \text{ for all } t > 0.$$

Hence f has an absolute maximum at $t = 1/(\sqrt{2}c)$.