Cal Poly Department of Mathematics

Puzzle of the Week

Feb 27 - Mar 12, 2015

From Tom O'Neil:

Find all ordered pairs of real numbers (x, y) with the property that the sum of their reciprocals is -1 and the sum of their cubes is 4.

Solutions should be submitted to Morgan Sherman:

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before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> There are two such ordered pairs: $(x, y) = (\frac{1\pm\sqrt{5}}{2}, \frac{1\mp\sqrt{5}}{2})$.

From $\frac{1}{x} + \frac{1}{y} = -1$ we have $x + y = -xy$. Let $z = xy$. Then we see that

$$-(xy)^3 = (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = 3xy(x+y) + (x^3 + y^3) = -3(xy)^2 + 4,$$

so that if $z = xy$ then z satisfies

$$z^3 - 3z^2 + 4 = 0.$$

The left side factors as $(z+1)(z-2)^2$ and therefore we have either $z = -1$ or $z = 2$. In the latter case we find $y = 2/x$ and solving for x we find $x = \frac{1}{2}(-2 \pm \sqrt{-4})$ which is not real. Hence we must have $z = -1$. So $y = -1/x$ and we find $\frac{1}{x} - x = -1$, which gives

$$x = \frac{1 \pm \sqrt{5}}{2},$$

and the solutions above follow.

Note: The number $\varphi = \frac{1+\sqrt{5}}{2}$ is known as the *golden ratio* and satisfies $\frac{-1}{\varphi} = \frac{1-\sqrt{5}}{2}$.