Cal Poly Department of Mathematics

Puzzle of the Week

Feb 13 - Feb 26, 2015

This one is a two-parter:

1. Find all positive integers equal to the sum of the squares of the digits of that number.
2. Find all positive integers equal to the square of the sum of the digits of that number.

Solutions should be submitted to Morgan Sherman:

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before the due date above. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> For (1) the only solution is the number 1, while for (2) the only solutions are 1 and 81.

For (1) let N be a solution. Note that N cannot have 4 or more digits, since, for example, the number 9999 has as its sum of squares of digits only 324, which is only three digits. So we may write $N = 100x + 10y, +z$ with $0 \leq x, y, z, \leq 9$. Then we have the equation $100x + 10y + z = x^2 + y^2 + z^2$ which implies

$$(100 - x)x + (10 - y)y = z(z - 1)$$

So x must equal 0, or else the number on the LHS will be at least 99, while the number on the RHS can be at most $9 \cdot 9 = 72$. Then we have $(10 - y)y = z(z - 1)$. From here one checks by hand that the set $\{y, 10 - y\}$ is not any of $\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5, 5\}$. Finally then we find $z = 1$ and get the answer above.

For (2) let N again denote a solution. We similarly first argue that N cannot have 5 or more digits, by considering the extreme case. So as above we have an equation

$$1000w + 100x + 10y + z = (w + x + y + z)^2$$

If $w \geq 2$ then the LHS is ≥ 2000 but the RHS is $\leq (9+9+9+9)^2 = 1296$, a contradiction. So $w = 0$ or 1. If $W = 1$ then the LHS is ≥ 1000 but the RHS is $\leq (1+9+9+9)^2 = 784$, another contradiction. So $w = 0$ and we have

$$100x + 10y + z = (x + y + z)^2$$

Let $s = x + y + z$. Then we can rewrite this as

$$9(11x + y) = s(s - 1)$$

So either s or $s - 1$ is divisible by 9, and since $1 \leq s \leq 27$ we have six cases: $s = 1, 9, 10, 18, 19, 27$. The first two lead to the answers $N = 1$ and $N = 81$ respectively, while the last four can be eliminated as leading to no solutions by a case by case analysis.