Cal Poly Department of Mathematics

Puzzle of the Week

Jan 30 - Feb 11, 2015

A large marble of radius r is dropped into a conical drinking cup and comes to rest tangent to the cup. A larger marble of radius $R > r$ is then dropped in and, as it turns out, is just touching the first marble as soon as it nestles into the cup (so the two spheres are tangent to each other, and each one is tangent to the cone in a circle of latitude). Find the volume of the region that lies between the two marbles and inside the cup.

Solutions should be submitted to Morgan Sherman:

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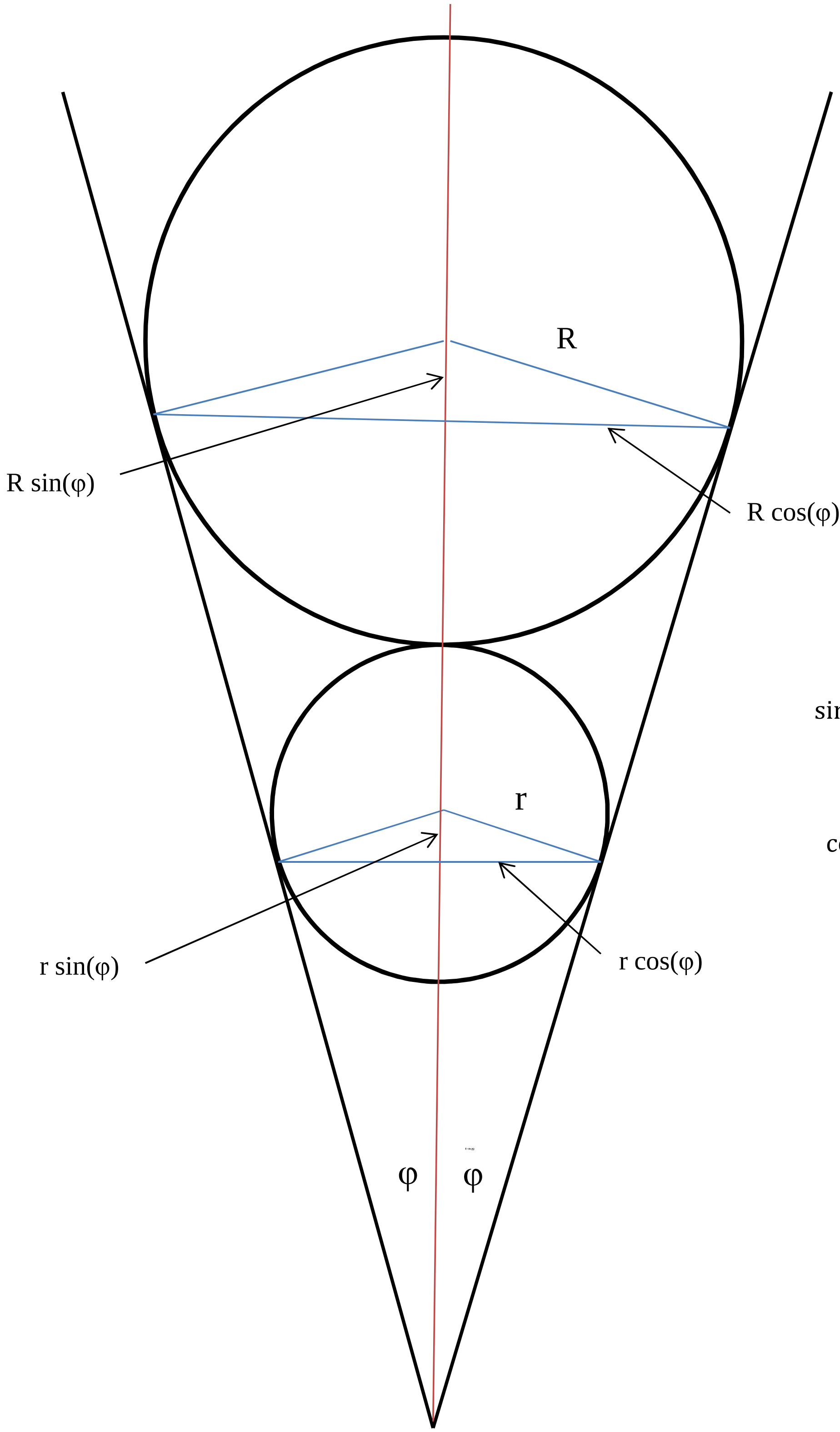
before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: I shamelessly stole this problem from the 1948 Putnam exam. So you shouldn't feel too bad if you tried it and couldn't get it. I'm embarrassed to admit how much scratch paper I used in getting it myself. I received a nice detailed solution, with a nice image, from retired professor Tom O'Neil, who has nicely let me use his work.

So, the next two pages is due to Tom O'Neil:



$$\sin(\varphi) = \frac{R - r}{R + r}$$

$$\cos(\varphi) = \frac{2\sqrt{Rr}}{R + r}$$

We seek the volume of the region between two adjacent spheres resting in a cone. Recall the volume of a frustum of a spherical cone is

$$V_c = \frac{\pi}{3} h(R_1^2 + R_1 R_2 + R_2^2)$$

and that of a polar cap of a sphere is

$$V_p = \frac{\pi}{3} h^2(3r - h).$$

In the accomping figure one can use a lot of geometry and trigometric identities to attain the values of the trignometric functions in terms of the sphere radii. We have

$$\sin(\varphi) = \frac{R-r}{R+r} \quad \text{and} \quad \cos(\varphi) = \frac{2\sqrt{rR}}{R+r}$$

Dealing with the cone first we have a height of

$$h = [R - R\sin(\varphi)] + [r + r\sin(\varphi)] = \frac{2rR}{R+r} + \frac{2rR}{R+r} = \frac{4rR}{R+r}$$

and since $R_1 = R\cos(\varphi)$ and $R_2 = r\cos(\varphi)$ we have

$$R_1^2 = R^2\left(\frac{4rR}{(R+r)^2}\right) = \frac{4rR^3}{(R+r)^2},\ R_2^2 = r^2\left(\frac{4rR}{(R+r)^2}\right) = \frac{4r^3R}{(R+r)^2} \text{ and } R_1\,R_2 = \frac{4r^2R^2}{(R+r)^2}$$

so

$$V_c = \frac{\pi}{3}\left(\frac{4rR}{R+r}\right)\left(\frac{4rR^3}{(R+r)^2} + \frac{4r^2R^2}{(R+r)^2} + \frac{4r^3R}{(R+r)^2}\right) = \frac{\pi}{3}\frac{16r^2R^2}{(R+r)^3}(R^2 + rR + r^2).$$

The larger top sphere has a volume of

$$V_R = \frac{\pi}{3}\left(\frac{2rR}{R+r}\right)^2\left(3R - \frac{2rR}{R+r}\right) = \frac{\pi}{3}\frac{4r^2R^2}{(R+r)^3}(3R^2 + rR)$$

and the smaller

$$V_r = \frac{\pi}{3}\left(\frac{2rR}{R+r}\right)^2\left(3r - \frac{2rR}{R+r}\right) = \frac{\pi}{3}\frac{4r^2R^2}{(R+r)^3}(3r^2 + rR).$$

The sum of the polar cap volumes is

$$V_R + V_r = \frac{\pi}{3}\frac{4r^2R^2}{(R+r)^3}(3R^2 + 2rR + 3r^2).$$

Subtracting this from the volume of the cone will give us the value we are after. This is

$$\frac{\pi}{3}\frac{4r^2R^2}{(R+r)^3}(4R^2 + 4rR + 4r^2) - \frac{\pi}{3}\frac{4r^2R^2}{(R+r)^3}(3R^2 + 2rR + 3r^2)$$

$$= \frac{\pi}{3}\frac{4r^2R^2}{(R+r)^3}(R^2 + 2rR + r^2) = \frac{\pi}{3}\frac{4r^2R^2}{(R+r)^3}(R+r)^2 = \frac{\pi}{3}\frac{4r^2R^2}{R+r}.$$